Improper Integrals

<u>Introduction</u># The Riemann's integral $\int_a^b f(x)dx$ or Riemann Stieltjes integral $\int_a^b f d\alpha$ is defined under the restrictions that f, is bounded in [a, b] and defined in in finite interval [a b]. If any one or all the conditions are relaxed the integral is called improper or infinite or generalised integrals.

Improper Integral of 1st Kind

The integral $\int_a^b f(x)dx$ is said to be improper integral of 1st kind if one or both of integration limits are infinite. i.e It will be of form.

$$\int_a^{\infty} f(x)dx \text{ or } \int_{-\infty}^{b} f(x)dx \text{ or } \int_{-\infty}^{+\infty} f(x)dx$$

Improper Integral of 2nd kind

The integral $\int_a^b f(x)dx$ is said to be improper integral of 2nd kind if f, is unbounded at finite no of points of infinite discontinuity in bounded interval [a, b]

Improper Integral of 3rd kind or mixed type

An improper integral $\int_a^b f(x)\,dx$ is said to be of 3rd kind or of mixed type if f is unbounded at finite no of points of integration. i.e it has unbounded interval of integration. i.e it has mixed conditions of 1st kind and 2nd kind

e.g

$$\int_0^\infty \frac{1}{x}\,dx, \quad \int_{-\infty}^{\infty} \frac{1}{x^2}\,dx, \quad \int_1^\infty \frac{1}{\sqrt{1-x^2}}\,dx$$

Note # The word bounded in improper integral of 1st kind seems to be redundant. But some authors extend the class of functions integrable in Riemann sense to include those unbounded functions whose improper integral exist

Convergence of Improper Integral of first kind (Infinite range of integration)

Convergence at ∞

Let f be bounded and integrable in $[a\ t]$, $\forall\, t \geq a$ i.e $t \in [a\ \infty[$ so that the proper integral $\int_a^t f(x)\,dx$ exist and is a function of variable $t \in [a\ \infty[$ we put

$$\varphi(t) = \int_a^t f(x)\,dx.$$

φ is a function with domain $[a\ \infty[$ If $\lim_{t\to\infty} \varphi(t)$ exist, then improper integral $\int_a^\infty f(x)\,dx$ exists or Converges at ∞ and regard symbol $\int_a^\infty f(x)\,dx$ as denoting the limit. Thus by definition

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

provided the limit exists

Convergence at $-\infty$

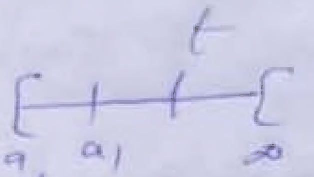
Let f be bounded and integrable in $[t, b]$ $\forall\, t \le b$ i.e $t \in]-\infty, b]$ so that proper integral $\int_t^b f(x)\,dx$ exists and is a function of t. we put

$$\varphi(t) = \int_t^b f(x)\,dx$$

φ is a function with domain $]-\infty\ b]$ If $\lim_{t\to-\infty} \varphi(t)$ exists, then improper integral $\int_{-\infty}^b f(x)\,dx$ exists or Converges at $-\infty$

and $\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty}\phi(t) = \lim_{t\to-\infty}\int_{t}^{b} f(x)\,dx$

provided limit exists



<u>Note</u> # (1) For $a_1 > a$

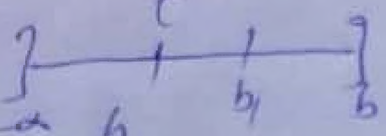
$$\int_a^t f(x)\,dx = \int_a^{a_1} f(x)\,dx + \int_{a_1}^{t} f\,dx \qquad t \geq a_1 > a$$

$\because \int_a^{a_1} f(x)\,dx$ is proper and hence cgt

$\therefore$ Convergence or divergence of $\int_a^{\infty} f(x)\,dx$ depends upon convergence or divergence of $\int_{a_1}^{\infty} f(x)\,dx$

$\Rightarrow$ Integrals $\int_a^{\infty} f(x)\,dx$ & $\int_{a_1}^{\infty} f(x)\,dx$ are either both cgt or both divergent

Thus when testing $\int_a^{\infty} f(x)\,dx$ for convergence we can replace it by $\int_{a_1}^{\infty} f(x)\,dx$ for any convenient $a_1 > a$



(2) For $b_1 < b$

$$\int_t^b f(x)\,dx = \int_t^{b_1} f(x)\,dx + \int_{b_1}^{b} f(x)\,dx \qquad t \leq b_1 <$$

$\because \int_{b_1}^{b} f(x)\,dx$ is proper

$\therefore$ Convergence or divergence of $\int_{-\infty}^{b} f(x)dx$ depends upon convergence or divergence of $\int_{-\infty}^{b_1} f(x)\,dx$

$\Rightarrow$ Integrals $\int_{-\infty}^{b} f(x)dx$, $\int_{-\infty}^{b_1} f(x)\,dx$ either both converge or both diverge

Therefore when testing $\int_{-\infty}^{b} f(x)dx$ for convergence we can replace it by $\int_{-\infty}^{b_1} f(x)\,dx$ for any convenient $b_1 < b$

Convergence at both ends $]-\infty\ +\infty[$

If c is any point in $]-\infty, +\infty[$ and $\int_{-\infty}^{c} f(x)dx$, $\int_{c}^{+\infty} f(x)dx$ both converge, then $\int_{-\infty}^{+\infty} f(x)dx$ exists and we write

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{\infty} f(x)dx$$

It is independent of the choice of c i.e. we can take convenient c in $]-\infty\ +\infty[$

Principal Value

$\lim_{t\to\infty}\int_{-t}^{t} f(x)\,dx$ is called principal val of improper integral $\int_{-\infty}^{+\infty} f(x)\,dx$

$$P.\int_{-\infty}^{+\infty} f(x)\,dx = \lim_{t\to\infty}\int_{-t}^{t} f(x)\,dx$$

provided limit exists

Note $\lim_{t\to\infty}\int_{t}^{a} f(x)\,dx + \lim_{t\to\infty}\int_{a}^{t} f(x)\,dx$

is not equal to $\lim_{t\to\infty}\left[\int_{-t}^{a} f(x)\,dx + \int_{a}^{t} f(x)\,dx\right.$

If $\int_{-\infty}^{+\infty} f(x)\,dx$ exists, then value of the integral equals principal value other wise principal value may exists even if $\int_{-\infty}^{+\infty} f(x)\,dx$ does not converge. e.g

Consider $\int_{-\infty}^{+\infty} x e^{x^2}$

$$\lim_{t\to-\infty}\int_{t}^{0} x e^{x^2}\,dx = \frac{1}{2}\lim_{t\to-\infty}\left[e^{x^2}\right]_{t}^{0}$$

$$= \frac{1}{2}\lim_{t\to-\infty}\left[1-e^{t^2}\right] = -\infty$$

So $\int_{-\infty}^{+\infty} x e^{x^2}$ does not exist but

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$$\lim_{t\to\infty}\int_{-t}^{t} x e^{x^2} = \frac{1}{2}\lim_{t\to\infty}\left[e^{x^2}\right]_{-t}^{t}$$

$$= \frac{1}{2}\lim_{t\to\infty}\left[e^{t^2} - e^{t^2}\right] = 0$$

Analogy Between Improper Integral and Infinite Series

$$\int_a^\infty \longleftrightarrow \sum_{n=1}^{\infty}$$

$$f(x) \longleftrightarrow a_n = f(n)$$

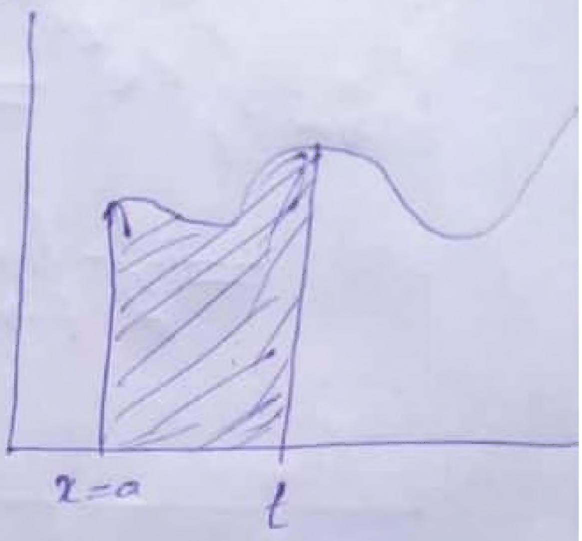
$$\int_a^t f(x)\,dx \longleftrightarrow \sum_{k=1}^{n} a_k$$

partial sum

Geometric Interpretation

Geometrically improper integral represent area under the curve which could be infinite. For integral $\int_a^\infty f(x)\,dx$ Consider partial sum $\int_a^t f(x)\,dx$, which represents area under curve $y = f(x)$

from a, to t,
When t increases
the area becomes
greater and greater.
If the area under
curve is finite when



$t \to +\infty$, then $\int_a^{\infty} f(x)$ is cgt and give area otherwise it is dgt. We note that if $f(x)$ rises (increases), then area increases without any bound and becomes infinite. Thus $\int_a^{\infty} f(x)dx$ may ~~will~~ converge if f decreases as $x \to \infty$ and touches x-axis i.e $\lim_{x \to \infty} f(x) = 0$ i.e function dies at infinity.

Result# Convergence of $\int_a^{\infty} f(x)dx \Rightarrow \lim_{x \to \infty} f(x) = 0$

But the converse may not be true. But

If $\lim_{x \to \infty} f(x) \neq 0$, then $\int_a^{\infty} f(x)\,dx$ is dgt

This is just like nth term divergence test for infinite series i.e if $\lim_{n \to \infty} a_n \neq 0$, then series $\sum_{n=1}^{\infty} a_n$ is dgt.

Example

Check the Convergence of (i) $\int_1^\infty \frac{1}{x}\,dx$ (ii) $\int_1^\infty \frac{1}{x^2}\,dx$

Sol (i) $\int_1^t \frac{1}{x}\,dx = [\ln x]_1^t = \ln t - \cancel{\ln 1}^{\,0}$

$$\lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln t = \infty$$

$\Rightarrow \int_1^\infty \frac{1}{x}\,dx$ is divergent

(ii) $$\int_1^t \frac{1}{x^2}\,dx = -\left[\frac{1}{x}\right]_1^t$$
$$= -\left[\frac{1}{t} - 1\right]$$

$$\lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx = -\lim_{t\to\infty}\left[\frac{1}{t} - 1\right]$$
$$= -[0-1] = 1, \text{ finite}$$

$\Rightarrow \int_1^\infty \frac{1}{x^2}\,dx$ is cgt.

Note note that $\lim_{x\to\infty} \frac{1}{x} = 0 = \lim_{x\to\infty}\frac{1}{x^2} = 0$

But $\int_1^\infty \frac{1}{x}\,dx$ is dgt & $\int_1^\infty \frac{1}{x^2}\,dx$ is cgt i.e

If $\lim_{x\to\infty} f(x) = 0$, then $\int_a^\infty f(x)\,dx$ may converge or diverge but if $\lim_{x\to\infty} f(x) \neq 0$, then $\int_a^\infty f(x)\,dx$ is not cgt. Thus $\lim_{x\to\infty} f(x) = 0$ is necessary Condition but not sufficient for the convergence

Example Examine for Convergence

(i) $\int_0^{\infty} \sin x\,dx$ (ii) $\int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$ (iii) $\int_2^{\infty} \frac{2x^2\,dx}{x^4-1}$

(iv) $\int_{-\infty}^{+\infty} \frac{dx}{(x^2+1)^2}$ (v) $\int_0^{\infty} x^3 e^{-x^2} dx$

(vi) $\int_0^{\infty} \sin 2\pi x\,dx$

Solutions (i) $\int_0^{\infty} \sin x\,dx$

$$\int_0^t \sin x\,dx = -[\cos x]_0^t = 1-\cos t$$

$$\lim_{t\to\infty}\int_0^t \sin x\,dx = \lim_{t\to\infty}(1-\cos t)$$

$$= 1-\lim_{t\to\infty}\cos t$$

$\because \lim_{t\to\infty}\cos t$ does not exist

$\therefore \int_0^{\infty} \sin x\,dx$ does not converge

(ii)

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$$

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{1}{1+x^2}dx + \int_0^{\infty} \frac{1}{1+x^2}dx$$

$$= \int_0^{\infty} \frac{1}{1+x^2}dx + \int_0^{\infty} \frac{1}{1+x^2}dx$$

$$= 2\int_0^{\infty} \frac{1}{1+x^2}dx$$

P.T.O

$$= 2\lim_{t\to\infty}\int_0^t \frac{1}{1+x^2}\,dx = 2\lim_{t\to\infty}\left[\tan^{-1}x\right]_0^t$$

$$= 2\lim_{t\to\infty}\left[\tan^{-1}t - \tan^{-1}0\right]$$

$$= 2\tan^{-1}(\infty) = 2(\pi/2) = \pi$$

$\Rightarrow$ Integral Converges to π

__An Explanation__

In $\displaystyle\int_{-\infty}^{0}\frac{1}{1+x^2}\,dx$ let $x = -y$

Then $dx = -dy$

__Limits__

When $x = -\infty$, $y = \infty$

$x = 0$ $y = 0$

$$= \int_{\infty}^{0}\frac{1}{1+y^2}(-dy)$$

$$= -\int_{0}^{\infty}\frac{1}{1+y^2}\,dy = \int_0^\infty \frac{1}{1+y^2}\,dy \quad \text{change of limits}$$

$$= \int_0^\infty \frac{1}{1+x^2}\,dx$$

change of variable does not not affect the value

If integrand is an even function, then in general

$$\star \quad \int_{-\infty}^{0} f(x)\,dx = \int_0^\infty f(x)\,dx$$

$\star$ note This noteable result with open eyes and keep it in your- - - - - - - -

OR

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \lim_{\substack{t_1 \to -\infty \\ t_2 \to +\infty}} \int_{t_1}^{t_2} \frac{dx}{1+x^2}$$

$$= \lim_{\substack{t_1 \to -\infty \\ t_2 \to +\infty}} \left[\operatorname{Tan}^{-1} x \right]_{t_1}^{t_2}$$

$$= \lim_{\substack{t_1 \to -\infty \\ t_2 \to +\infty}} \left(\operatorname{Tan}^{-1} t_2 - \operatorname{Tan}^{-1} t_1 \right)$$

$$= \operatorname{Tan}^{-1}(\infty) - \operatorname{Tan}^{-1}(-\infty)$$

$$= \bar{\pi}/2 + \bar{\pi}/2 = \bar{\pi}$$

(iii)

$$\int_2^{\infty} \frac{2x^2\,dx}{x^4-1}$$

$$\int_2^{\infty} \frac{2x^2\,dx}{x^4-1} = \lim_{t \to \infty} \int_2^{t} \frac{2x^2\,dx}{x^4-1}$$

$$= \lim_{t \to \infty} \int_2^{t} \left[\frac{1}{x^2-1} + \frac{1}{x^2+1} \right] dx \quad \text{(partial fractio}$$

$$= \lim_{t \to \infty} \left[\frac{1}{2} \ln \left| \frac{x-1}{x+1} \right| + \operatorname{Tan}^{-1} x \right]_2^{t}$$

$$= \lim \left[\frac{1}{2} \ln \left| \frac{t-1}{t+1} \right| - \frac{1}{2} \ln \frac{1}{3} + \operatorname{Tan}^{-1} t - \operatorname{Tan}^{-1} 2 \right] \; p$$

$$= \frac{1}{2} \lim_{t\to\infty} \ln\left(\frac{t-1}{t+1}\right) - \frac{1}{2}\ln 3^{-1} + \tan^{-1}\infty - \tan^{-1}2$$

mod deleted because $t \to \infty$ from $t-1 > 0$, $t+1 > 0$

$$= \frac{1}{2} \ln\left\{\lim_{t\to\infty}\left(\frac{t-1}{t+1}\right)\right\} + \frac{1}{2}\ln 3 + \frac{\pi}{2} - \tan^{-1}2$$

$$= \frac{1}{2}\cancel{\ln(1)}^{0} + \frac{1}{2}\ln 3 + \frac{\pi}{2} - \tan^{-1}2$$

$$= \frac{\pi}{2} - \tan^{-1}2 + \frac{1}{2}\ln 3$$

$\Rightarrow$ Integral is cgt

<u>Note</u> $\lim_{t\to\infty} \frac{t-1}{t+1} \quad \left(\frac{\infty}{\infty}\right) = \lim_{t\to\infty} \frac{1}{1} = 1$ Hospital rule

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(iv)

$$\int_{-\infty}^{+\infty} \frac{dx}{(1+x^2)^2}$$

$$I = \int \frac{dx}{(1+x^2)^2}$$ putting $x = \tan\theta$, $dx = \sec^2\theta\, d\theta$

$$= \int \frac{\sec^2\theta\, d\theta}{(1+\tan^2\theta)^2} = \int \frac{\sec^2\theta\, d\theta}{\sec^4\theta}$$

$$= \int \frac{1}{\sec^2\theta} d\theta = \int \cos^2\theta\, d\theta$$

$$= \int (1+\cos 2\theta)\, d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta$$

$$= \frac{1}{2}\theta + \frac{1}{4}\cdot\frac{2\tan\theta}{1+\tan^2\theta}$$ $\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}$

<u>P.T.O</u>

$$= \frac{1}{2}\tan^{-1}x + \frac{x}{2(1+x^2)}$$

Now

$$\int_{-\infty}^{+\infty}\frac{dx}{(1+x^2)^2} = \int_{-\infty}^{0}\frac{dx}{(1+x^2)^2} + \int_{0}^{+\infty}\frac{dx}{(1+x^2)^2}$$

$$= \int_{0}^{\infty}\frac{dx}{(1+x^2)^2} + \int_{0}^{\infty}\frac{dx}{(1+x^2)^2}$$

$$= 2\int_{0}^{\infty}\frac{dx}{(1+x^2)^2}$$

$$= 2\lim_{t\to\infty}\int_{0}^{t}\frac{dx}{(1+x^2)^2}$$

$$= 2\lim_{t\to\infty}\left[\frac{1}{2}\tan^{-1}x + \frac{x}{2(1+x^2)}\right]_0^t$$

$$= \lim_{t\to\infty}\left[\tan^{-1}t + \frac{t}{t^2+1}\right]$$

$$= \tan^{-1}(\infty) + \lim_{t\to\infty}\frac{t}{t^2+1} = \frac{\pi}{2} + 0 = \frac{\pi}{2}$$

$\Rightarrow$ Integral converges to $\frac{\pi}{2}$

(V)

$$\int_{0}^{\infty} x^3 e^{-x^2}\,dx$$

$$I = \int x^3 e^{-x^2}\,dx = -\frac{1}{2}\int x^2(-2x)e^{-x^2}\,dx$$

$$= -\frac{1}{2}\left[x^2 e^{-x^2} - \int e^{-x^2}\cdot 2x\,dx\right]$$

$$= -\frac{1}{2}x^2 e^{-x^2} - \frac{1}{2}\int(-2x)\,e^{-x^2}\,dx$$

(15)

$$I = -\frac{1}{2}x^2 e^{-x^2} - \frac{1}{2}\int -2x\, e^{-x^2} dx$$

$$= -\frac{1}{2}x^2 e^{-x^2} - \frac{1}{2} e^{-x^2} = -\frac{1}{2}(x^2+1)e^{-x^2}$$

$$\int_0^\infty x^3 e^{-x^2} dx = \lim_{t\to\infty} \int_0^t x^3 e^{-x^2} dx$$

$$= \lim_{t\to\infty} \left[-\frac{1}{2}(x^2+1)e^{-x^2}\right]_0^t$$

$$= \lim_{t\to\infty} \left[\frac{1}{2} - \frac{1}{2}(t^2+1)e^{-t^2}\right]$$

$$= \frac{1}{2} - \frac{1}{2}\lim_{t\to\infty} \frac{t^2+1}{e^{t^2}} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \frac{1}{2} - \frac{1}{2}(0) = \frac{1}{2}$$

$\Rightarrow$ Integral Converges

(Vi)

$$\int_0^\infty \sin 2\pi x\, dx$$

$$\int_0^\infty \sin 2\pi x\, dx = \lim_{t\to+\infty} \int_0^t \sin 2\pi x\, dx$$

$$= \lim_{t\to\infty} \left[-\frac{1}{2\pi}\cos 2\pi x\right]_0^t$$

$$= -\frac{1}{2}\lim_{t\to\infty}\left[\cos 2\pi t - 1\right] \Rightarrow \lim_{t\to\infty}\cos 2\pi t \text{ does not exist}$$

$\Rightarrow$ Integral is divergent

Review:-

→ A function f is said to be increasing if for all $x_1, x_2 \in D_f$ (domain of f) and $x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2)$

→ A function f is said to be bounded if there exist some +ve number M such that
$$|f(x)| \leq M \quad \forall x \in D_f$$

→ If f is defined on $[a\ +\infty)$ and $\lim_{x\to\infty} f(x)$ exists, then f is bounded on $[a\ +\infty)$

→ If $f \in R[a,b]$ & $c \in (a,b)$, then
$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

→ If $f \in R[a,b]$ and $f(x) \geq 0 \quad \forall x \in [a,b]$, then
$$\int_a^b f(x)dx \geq 0$$

→ If f is monotonically increasing on $[a\ +\infty)$ and bounded on $[a\ +\infty)$, then $\lim_{x\to\infty} f(x) = \sup_{x\in[a\ +\infty)} f(x)$

→ If $f, g \in R[a,b]$ and $f(x) \leq g(x)$ $\forall x \in [a,b]$, then
$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

☆ Patience, hardworking, motivation, clear destination and trust on God are mile stones for success. Ponder over it?

☆ Be courageous enough to review you & your learning

Convergence Test at $+\infty$

(Integrand retaining sign in interval of integration)

There is no loss of generality if we suppose that f is +ve in $[a, +\infty)$ because if f is -ve, then it can be replaced by $-f = g$, which is +ve in $[a, +\infty)$. Also tests are given for cases where integration limit is $+\infty$. Similar test exist where integration limit is $-\infty$ (a change of variable $x = -y$ then makes the integration limit $+\infty$)

Theorem# Suppose that $f \in R[a, t]$, $\forall t \geq a$ i.e f is integrable in $[a, \infty)$ and $f(x) \geq 0$ $\forall x \geq a$. Then $\int_a^\infty f\,dx$ Converges iff there exists a no $M > 0$ such that

$$\int_a^t f(x)dx \leq M \quad \forall t \geq a$$

i.e partial sum $\int_a^t f(x)dx$ is bounded above

Proof# As $f(x) \geq 0$, $\forall x \in [a, t]$ $\forall t \geq a$

Therefore $I(t) = \int_a^t f(x)dx \geq 0$ $\forall t \geq a$

Suppose that $\int_a^\infty f(x)dx$ is convergent. Then $\lim_{t \to \infty} I(t)$ exists and hence $I(t)$ is bounded on $[a, +\infty)$. So $\exists$ a no $M > 0$ s.t

$|I(t)| \leq M \quad \forall t \geq a$

$\Rightarrow I(t) \leq M \quad \forall t \geq a \quad \because I(t) \geq 0 \quad \forall t \geq a$

$\Rightarrow \int_a^t f(x)dx \leq M \quad \forall t \geq a$

Converse# Conversely suppose that $\exists$ a no. M such that

$$\int_a^t f(x)dx \leq M \quad \forall t \geq a$$

$$\Rightarrow |I(t)| \leq M \quad \forall t \geq a$$

$\Rightarrow I(t)$ is bounded on $[a, +\infty)$

Now for $t_2 \geq t_1 > a$, we have

$$I(t_2) = \int_a^{t_2} f(x)dx = \int_a^{t_1} f(x)\,dx + \int_{t_1}^{t_2} f(x)\,dx$$

$$\geq \int_a^{t_1} f(x)dx = I(b_1)$$

$\because \int_{t_1}^{t_2} f(x)\,dx \geq 0$ as $f(x) \geq 0 \quad \forall x \geq a$

$\Rightarrow I(t)$ is monotonically increasing on $[a, +\infty)$. As $I(t)$ is monotonically increasing and bounded on $[a, +\infty)$, Therefore $\lim_{t\to\infty} I(t)$ exists

i.e $\int_a^\infty f\,dx$ converges

OR

It can be proved as

$\because I(t) \geq 0 \quad \forall t \geq a$

$\therefore I(t)$ is monotonically increasing as t increases

$\Rightarrow$ $I(t)$ will tend to finite limit
iff it is bounded above
iff $\exists$ a +ve no M such that
$|I(t)| \leq M \quad t \geq a$
iff $I(t) \leq M \quad \forall t \geq a$
iff $\int_a^t f(x)\,dx \leq M \quad \forall t \geq a$

Thus $\int_a^\infty f(x)\,dx$ will be cgt iff $\int_a^t f(x)\,dx \leq M \quad \forall t \geq a$

Comparison Test

(Integrands +ve on interval of integration)

If f, g are integrable in $[a \; +\infty)$
and $0 \leq f(x) \leq g(x) \quad \forall x \in [a \; +\infty)$.
Then
(i) $\int_a^\infty g(x)\,dx$ cngt $\Rightarrow$ $\int_a^\infty f(x)$ is Convergent
(ii) $\int_a^b f(x)\,dx$ dgt $\Rightarrow$ $\int_a^\infty g(x)$ is divergent

Proof# $\because \; 0 \leq f(x) \leq g(x) \quad \forall x \geq a$

$$\therefore \int_a^t f(x)\,dx \leq \int_a^t g(x) \quad \forall t \geq a \longrightarrow (1)$$

(i) $\int_a^\infty g(x)\,dx$ be cgt Then $\exists$ a +ve no M
such that. P.T.O

$$\int_a^t g(x)dx \le M \quad \forall t \geq a \longrightarrow (2)$$

By (1) & (2)

$$\int_a^t f(x)dx \le \int_a^t g(x)dx \le M \quad \forall t \geq a$$

$\Rightarrow \int_a^\infty f(x)dx$ is cgt

(ii) Let $\int_a^\infty f(x)d$ be divergent. Then $\int_a^t f(x)dx$ is unbounded above and by (1) $\int_a^t g(x)dx$ is also unbounded above and hence $\int_a^\infty g(x)dx$ is dgt

Limit Test (Comparison)

Suppose that $f(x) \geq 0, g(x) \geq 0 \quad \forall x \in [a \ \infty)$ and $f, g \in R[a \ t] \quad \forall t \geq a$. Then

(i) if $\lim_{x \to \infty} \frac{f(x)}{g(x)} = l \qquad 0 < l < \infty$, then $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ behave alike

(ii) If $l = 0$ and $\int_a^\infty g(x)dx$ is cgt, then $\int_a^\infty f(x)dx$ is cgt

(iii) if $l = \infty$ and $\int_a^\infty g(x)dx$ is dgt, then $\int_a^\infty f(x)dx$ is dgt

P.T.O

Proof # (i) $\because \lim_{x\to\infty} \frac{f(x)}{g(x)} = l$

$\therefore$ By definition of limit of function at ∞

For any $\epsilon > 0$, $0 < \epsilon < l$, we can find a number $N \geq a$, however Large such that

$$\left| \frac{f(x)}{g(x)} - l \right| < \epsilon \quad \forall x > N \geq a$$
$$0 < \epsilon < l, \; l - \epsilon > 0$$

$$\Rightarrow (l-\epsilon)\, g(x) \leq f(x) \leq (l+\epsilon)\, g(x) \quad \forall x > N \geq a \longrightarrow ①$$

if $\int_a^\infty f(x)dx$ Converges, then $\int_N^\infty f(x)dx$ also Converges and hence by Comparison test $\int_N^\infty g(x)dx$ and therefor $\int_a^\infty g(x)dx$ is cgt

If $\int_a^\infty f dx$ diverges, Then $\int_N^\infty f(x)dx$ also diverges and by Comparison test $\int_N^\infty g(x)dx$ and therefore $\int_a^\infty g(x)dx$ diverges

(ii)

$\because \lim_{x\to 8} \frac{f(x)}{g(x)} = 0$

$\therefore$ for any $\epsilon > 0$, we can find a no N such that $\quad \forall x > N \geq a$

$$\left| \frac{f(x)}{g(x)} - 0 \right| < \epsilon$$

$$\Rightarrow \left|\frac{f(x)}{g(x)}\right| < \epsilon \quad \forall x > N \geq a$$

$$\Rightarrow \frac{f(x)}{g(x)} < \epsilon \quad \text{"} \quad \text{"}$$

$\because f(x) \geq 0 \ \forall x$
$g(x) \geq 0$

$$\Rightarrow f(x) < \epsilon g(x) \quad \forall x > N$$

If $\int_a^\infty g\,dx$ Converges, then $\int_N^\infty g(x)\,dx$ also Converges and by Comparison test $\int_N^\infty f(x)\,dx$ and $\int_a^\infty f(x)\,dx$ Converges

(iii)

$$\because \lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$$

$\therefore$ For any no. M (hower Large), $\exists$ a no. N such That

$$\frac{f(x)}{g(x)} > M \quad \forall x \geq N \geq a$$

$$\Rightarrow f(x) > M g(x) \quad \text{"} \quad \text{"}$$

If $\int_a^\infty g(x)\,dx$ diverges, then $\int_N^\infty g(x)\,dx$ also diverges and by Comparison test $\int_N^\infty f(x)\,dx$ and hence $\int_a^\infty f(x)\,dx$ diverges

Rieview

* If $\lim_{x \to \infty} f(x) = l$, then for any $\epsilon > 0$, $\exists N > 0$ such that $|f(x) - l| < \epsilon \quad \forall x > N$
* If $\int_a^\infty f(x)$ converges (diverges), then $\int_N^\infty f(x)dx$ converges (diverges) for $N > a$ ~~if f is bounded in [a, N]~~
* If $\int_a^\infty f(x)dx$ converges (diverges), then $\int_N^\infty f(x)dx$ converges (diverges) for $N < a$ if f is bounded in $[N, a]$
* We say $\lim_{x \to a} f(x) = \infty$ if for any $M > 0$ $\exists \delta > 0$ such that
$$f(x) > M \quad \forall x, \ 0 < |x - a| < \delta$$
* We say that $\lim_{x \to a} f(x) = -\infty$ if for any $M > 0$, $\exists$ $\delta > 0$ such that
$$f(x) < -M \quad \forall x, \ 0 < |x - a| < \delta$$
* We say that $\lim_{x \to \infty} f(x) = \infty$ if for any $M > 0$ $\exists$ a no $N > 0$ such that
$$f(x) > M \quad \forall x > N$$
* We say that $\lim_{x \to \infty} f(x) = -\infty$ if for any $M > 0$ $\exists$ $N > 0$ such that
$$f(x) < -M \quad \forall x > N$$

* We say that $\lim_{x \to -\infty} f(x) = \infty$ if for any $M > 0$ $\exists$ a no $N > 0$ such that

$$f(x) > M \quad \forall x < -N$$

* We say that $\lim_{x \to -\infty} f(x) = -\infty$ if for any $M > 0$ $\exists$ a no $N > 0$ such that

$$f(x) < -M \quad \forall x < -N$$

Some useful Comparison integrals

(a) P- integral

$\int_a^\infty \frac{dx}{x^p}$ where p is a constant, $a > 0$ converges if $p > 1$ and diverges if $p \leq 1$

(b) Geometric or Exponential Integral

$\int_a^\infty e^{-\lambda x}$, where λ is a constant, converges if $\lambda > 0$ and diverges if $\lambda \leq 0$

Note the analogy with geometric series if $r = e^{-\lambda}$ so that $e^{-\lambda x} = r^x$

Proof# (a)

Case 1 when $p = 1$

$$\int_a^\infty \frac{dx}{x^p} = \int_a^\infty \frac{1}{x} dx$$

$$\int_a^t \frac{1}{x}\,dx = \Big[\ln x\Big]_a^t \quad \because x \in [a,\infty[,\ a>0 \quad \therefore \ln|x| = \ln x$$

$$= \ln\left(\frac{t}{a}\right)$$

$$\lim_{t\to\infty}\int_a^t \frac{1}{x}\,dx = \lim_{t\to\infty}\ln\left(\frac{t}{a}\right) = +\infty$$

$\Rightarrow$ Integral diverges for $P=1$

<u>Case II</u> When $P \neq 1$

$$\int_a^t f(x)\,dx = \int_a^t \frac{dx}{x^P} = \int_a^t x^{-P}\,dx$$

$$= \left[\frac{x^{-P+1}}{1-P}\right]_a^t$$

$$= \begin{cases} \left[\dfrac{x^{1-P}}{1-P}\right]_a^t = \dfrac{1}{1-P}\left[t^{1-P} - a^{1-P}\right] & \text{if } 1-P>0 \text{ i.e. } P<1 \\ -\dfrac{1}{P-1}\left[\dfrac{1}{t^{P-1}} - \dfrac{1}{a^{P-1}}\right] & \text{if } P-1>0 \text{ i.e. } P>1 \end{cases}$$

$$\Rightarrow \lim_{t\to\infty}\int_a^t \frac{dx}{x^P} = \begin{cases} \infty & \text{if } P<1 \\ \dfrac{1}{a^{P-1}(P-1)} & \text{if } P>1 \end{cases}$$

Thus $\int_a^\infty \frac{dx}{x^P}$ diverges for $P \leq 1$

and Converges for $P>1$

<u>Note</u> If C is +ve constant, then $\int_a^\infty \frac{c}{x^P}\,dx$, $a>0$ is cgt for $P>1$ and dgt for $P \leq 1$

(b) $\int_a^t e^{-\lambda x} dx = \left[\frac{e^{-\lambda x}}{-\lambda}\right]_a^t$

$= -\frac{1}{\lambda}\left[e^{-\lambda t} - e^{-\lambda a}\right]$

$$\lim_{t\to\infty}\int_a^t e^{-\lambda x} dx = \begin{cases} -\frac{1}{\lambda}\left[0 - e^{-\lambda a}\right] & \text{if } \lambda > 0 \\ = \frac{1}{\lambda} e^{-\lambda a}, \text{ finite} & a \text{ may be +ve or -ve} \\ +\infty & \text{if } \lambda < 0 \end{cases}$$

Thus $\int_a^\infty e^{-\lambda x} dx$ converges for $\lambda > 0$ and diverges for $\lambda < 0$ e.g. $\int_1^\infty e^{-1x} dx$ $\int_1^\infty e^{-2x} dx$ are convergent but $\int_1^\infty e^{x} dx = \int_1^\infty e^{-(-1)x} dx$ and $\int_1^\infty e^{2x} dx = \int_1^\infty e^{-(-2)x} dx$ are divergent

Deduction forom Limit Comparison Test

Taking $g(x) = \frac{1}{x^p}$, we have

If $\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty} x^p f(x) = A$, A is finite and $p >$

, then $\int_a^\infty f(x) dx$ is cgt because $\int_a^\infty g(x) dx$ is cgt

If $\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty} x^p f(x) = A$ ($A \neq 0$, it may be infinite) and $p \le 1$

, then $\int_a^\infty f(x) dx$ is dgt

Example

Test for convergence (a) $\displaystyle\int_1^\infty \frac{x\,dx}{3x^4+5x^2+1}$

(b) $\displaystyle\int_2^\infty \frac{x^2-1}{\sqrt{x^6+16}}\,dx$

Solution

(a) $\displaystyle\int_1^\infty \frac{x\,dx}{3x^4+5x^2+1}$

$$\frac{x}{3x^4+5x^2+1} \sim \frac{x}{3x^4} \quad \text{(Taking dominant Terms)}$$

$$\sim \frac{x}{x^4} \quad \text{(ignoring Constant multiples)}$$

$$= \frac{1}{x^3} \quad \text{(simplify)}$$

Now $\displaystyle\int_1^\infty \frac{1}{x^3}\,dx$ is cgt by p-integral

So $\displaystyle\int_1^\infty \frac{x\,dx}{3x^4+5x^2+1}$ is also cgt

We can compare it with $g(x) = \frac{1}{x^2}$ as

Let $f(x) = \dfrac{x}{3x^4+5x^2+1}$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^3}{3x^4+5x^2+1} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{1/x}{3+\frac{5}{x^2}+\frac{1}{x^4}} = \frac{0}{3} = 0$$

$\because \displaystyle\int_1^\infty g(x)\,dx$ is cgt $\therefore$ By L.C.T $\displaystyle\int_1^\infty f(x)\,dx$ is cgt

(b) 28

$$\int_2^\infty \frac{x^2-1}{\sqrt{x^6+16}}\,dx$$

$$f(x) = \frac{x^2-1}{\sqrt{x^6+16}} \sim \frac{x^2}{\sqrt{x^6}}$$ (Taking dominant Terms)

$$= \frac{x^2}{x^3} = \frac{1}{x}$$

Let $g(x) = \frac{1}{x}$. Then $\int_2^\infty \frac{1}{x}\,dx$ is dgt

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^3-x}{\sqrt{x^6+16}}$$

$$= \lim_{x\to\infty} \frac{x^3-x}{\sqrt{x^6\left(1+\frac{16}{x^6}\right)}}$$

$$= \lim_{x\to\infty} \frac{x^3\left(1-\frac{1}{x^2}\right)}{\sqrt{(x^3)^2}\sqrt{1+\frac{16}{x^6}}}$$

$$= \lim_{x\to\infty} \frac{x^3\left(1-\frac{1}{x^2}\right)}{|x^3|\sqrt{1+\frac{16}{x^6}}}$$ $\because \sqrt{x^2} = |x|$

$$= \lim_{x\to\infty} \frac{\cancel{x^3}\left(1-\frac{1}{x^2}\right)}{\cancel{x^3}\sqrt{1+\frac{16}{x^6}}}$$ as $x\to\infty$ so for very large x, x is +ve

$= 1$, non-zero finite

$\Rightarrow$ both integrals behave alike

$\because \int_2^\infty g(x)\,dx$ is dgt $\therefore \int_2^\infty f(x)\,dx$ is dgt

Example

$$\int_1^\infty e^{-x^2} dx$$

We can not evaluate integral explicitly

$\because \quad x^2 \geq x \quad \forall x \in [1, \infty[$

$\therefore \quad e^{-x^2} \leq e^{-x}$ (Exponentials with greater exponents are greater)

As $\int_1^\infty e^{-x} dx$ is cgt, Therefore by comparison test $\int_1^\infty e^{-x^2} dx$ is cgt

Example

$$\int_{1/2}^\infty e^{-x^2} dx$$

because $e^{-x^2} \leq e^{-x}$ is not true when $0 < x <$

In fact for $0 < x < 1$, $x^2 < x$ and $e^{-x^2} > e^{-x}$.

We write

$$\int_{1/2}^\infty e^{-x^2} dx = \int_{1/2}^1 e^{-x^2} dx + \int_1^\infty e^{-x^2} dx$$

First integral is proper and IInd integral converges as above. So given integral converges.

(30)

Example

Check for Convergence (a) $\int_1^\infty \frac{\sqrt{x}}{x^2+x}\,dx$

(b) $\int_1^\infty \frac{x+\sin x}{e^{-x}+x^2}\,dx$

Solutions

(a)

$$f(x) = \frac{\sqrt{x}}{x^2+x} \sim \frac{\sqrt{x}}{x^2} = \frac{1}{x^{3/2}}$$

Let $g(x) = \frac{1}{x^{3/2}}$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^2}{x^2+x} = 1, \text{ non-zero finite}$$

$\Rightarrow$ both integrals behave alike

$\because \int_1^\infty \frac{1}{x^{3/2}}\,dx$ is cgt $\therefore \int_1^\infty f(x)\,dx$ is cgt

(b)

When x is very large

- $e^{-x} \ll x^2$ so that $e^{-x}+x^2 \approx x^2$
- $|\sin x| \le 1 \ll x$ so $x+\sin x \approx x$
- The integrand $\frac{x+\sin x}{e^{-x}+x^2} \approx \frac{x}{x^2} = \frac{1}{x}$

Let $f(x) = \frac{x+\sin x}{e^{-x}+x^2}$ $\quad g(x) = \frac{1}{x}$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^2+x\sin x}{e^{-x}+x^2}$$

$$= \lim_{x\to\infty} \frac{1+\frac{\sin x}{x}}{e^{-x}/x^2+1} = \frac{1+0}{0+1} = 1$$

P.T.O

$\because \int_1^\infty g(x)\,dx$ is dgt $\therefore \int_1^\infty f(x)\,dx$ is dgt

Example

prove that, for every real p, the integral $\int_1^\infty e^{-x} x^p\,dx$ Converges

Solution

Let $f(x) = e^{-x} x^p = \dfrac{x^p}{e^x}$

let $g(x) = \dfrac{1}{x^n}$ we guess for value of n

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^p \cdot x^n}{e^x}$$

$$= \lim_{x\to\infty} \frac{x^n}{e^x} \quad \text{for } p = 0 \quad n > 0$$

$= 0$ from repeated applications of L-Hospital Rule

$$= \lim_{x\to\infty} \frac{x^n}{x^p e^x} = 0 \quad \text{for } p < 0, \; n > 0$$

$$= \lim_{x\to\infty} \frac{x^{p+n}}{e^x} = 0 \quad \forall p, n \text{ real}$$

So we must take such value of n for which $\int_1^\infty g(x)\,dx$ is cgt let $n = 2 \Rightarrow$

$$g(x) = \frac{1}{x^2} \Rightarrow \int_1^\infty \frac{1}{x^2}\,dx \text{ is cgt}$$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^{p+2}}{e^x} = 0 \quad \forall p \text{ real}$$

$\because \int_1^{\infty} g(x)\,dx$ is cgt $\therefore \int_1^{\infty} f(x)\,dx$ is cg
for every real p

<u>Note</u> * If $\int_a^{\infty} f(x)\,dx$ is cgt, then $\int_a^{\infty} c f(x)\,dx$ is cgt for every constant

* If $\int_a^{\infty} c f(x)\,dx$ is cgt for some non zero constant c, then $\int_a^{\infty} f\,dx$ is also cg

i.e removal of a non-zero constant or insertion of a non-zero constant does n affect the convergence and divergence

but if $\int_a^{\infty} f(x)\,dx$ is dgt, then $\int_a^{\infty} 0 f(x)\,dx$ is cgt i.e insertion of a zero constant makes dgt integral cgt and leaves cgt integral convergent

* $\int_a^{\infty} c\,dx$ is dgt for any non-zero constant

* $\int_0^{\infty} 0\,dx = 0$ is cgt

<u>Example</u>

Examine the convergence of

(i) $\int_0^{\infty} \frac{x\,dx}{(1+x)^3}$ (ii) $\int_1^{\infty} \frac{dx}{(1+x)\sqrt{x}}$

(iii) $\int_1^{\infty} \frac{dx}{x^{1/3}(1+x)^{1/2}}$ (iv) $\int_0^{\infty} \frac{\sin^2 x}{x^2}\,dx$

Solutions

(i) $f(x) = \frac{x}{(1+x)^3}$ take $g(x) = \frac{1}{x^2}$

(ii) $f(x) = \frac{1}{(1+x)\sqrt{x}}$ take $g(x) = \frac{1}{x^{3/2}}$

(iii) $f(x) = \frac{1}{x^{1/3}(1+x)^{1/2}}$ take $g(x) = \frac{1}{x^{5/6}}$

(iv)

Taking any $a > 0$

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \int_0^a \frac{\sin^2 x}{x^2}\,dx + \int_a^\infty \frac{\sin^2 x}{x^2}\,dx$$

$\int_0^a \frac{\sin^2 x}{x^2}\,dx$ is proper because $\lim_{x\to 0} \frac{\sin^2 x}{x^2} = 1$

so we check $\int_a^\infty \frac{\sin^2 x}{x^2}\,dx$

$$\because \sin^2 x \leq 1$$

$$\therefore \frac{\sin^2 x}{x^2} \leq \frac{1}{x^2}$$

Since $\int_a^\infty \frac{1}{x^2}\,dx$, $a > 0$ is cgt, therefore $\int_a^\infty \frac{\sin^2 x}{x^2}\,dx$ is also cgt by comparison test.

Thus $\int_0^\infty \frac{\sin^2 x}{x^2}\,dx$ is cgt

note that here we can not apply limit test because $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\sin^2 x}{x^2} \div \frac{1}{x^2}$

$= \lim_{x\to\infty} \sin^2 x$ does not exist

* Human being are also proper and improper. Some improper may converge at any stage of their life Ponder over it

Example

Examine the Convergence of

(i) $\int_1^\infty \frac{dx}{x\sqrt{x^2+1}}$ (ii) $\int_0^\infty \frac{x^2 dx}{\sqrt{x^5+1}}$

(iii) $\int_1^\infty e^{-x^2} dx$ (iv) $\int_1^\infty \frac{\log x}{x^2} dx$

Solutions

(i)

When $x \to \infty$ $\frac{1}{x\sqrt{x^2+1}} \sim \frac{1}{x\sqrt{x^2}} = \frac{1}{x^2}$

Let $f(x) = \frac{1}{x\sqrt{x^2+1}}$ $g(x) = \frac{1}{x^2}$

$\lim_{x\to\infty} \frac{f}{g}(x) = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = 1$, non-zero finite.

Hence $\int_1^\infty f(x)\,dx$ & $\int_1^\infty g(x)\,dx$ behave alike.

As $\int_1^\infty g(x)\,dx$ is cgt, therefore $\int_1^\infty f(x)\,dx$ is also cgt

(ii)

$f(x) = \frac{x^2}{\sqrt{x^5+1}} \sim \frac{x^2}{x^{5/2}} = \frac{1}{x^{1/2}}$

Let $g(x) = \frac{1}{x^{1/2}}$

$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^{5/2}}{\sqrt{x^5+1}} = \lim \frac{1}{\sqrt{1+\frac{1}{x^5}}} = 1$

$\because \int_1^\infty g(x)\,dx$ is dgt $\therefore \int_1^\infty f(x)\,dx$ is dgt

$\int_0^\infty e^{-x^2} dx$ (IInd Method)

0 is not point of infinite discontinuity and so we write

$$\int_0^\infty e^{-x^2} dx = \int_0^1 e^{-x^2} dx + \int_1^\infty e^{-x^2} dx$$

1st integral is proper

IInd integral $\int_1^\infty e^{-x^2} dx$

$$e^{x^2} > x^2 \quad \forall \text{ real } x$$

$$\Rightarrow \frac{1}{e^{x^2}} < \frac{1}{x^2} \quad \forall x \in [1, \infty[$$

$\because \int_1^\infty \frac{1}{x^2} dx$ is Convergent $\therefore \int_1^\infty e^{-x^2} dx$ is convergent and hence $\int_0^\infty e^{-x^2} dx$ is also cgt

(IV)

$$\int_1^\infty \frac{\log x}{x^2} dx$$

$f(x) = \frac{\log x}{x^2}$ let $g(x) = \frac{1}{x^p}$

we adjust value of p by limit comparison test as

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^p \log x}{x^2} = \lim_{x\to\infty} \frac{\log x}{x^{2-p}} \quad \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{1/x}{(2-p)x^{1-p}} = \lim_{x\to\infty} \frac{1}{2-p} \frac{1}{x^{2-p}} \to 0$$

if $2-p>0$ or $p<2$

We adjust $p < 2$ so that $\int_1^\infty \frac{1}{x^p}dx$ is cgt

Taking $p = 3/2 = 1.5 < 2$

$g(x) = 1/x^{3/2}$

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{\log x}{x^{1/2}} = \lim_{x\to\infty}\frac{1/x}{1/2\, x^{-1/2}}$$

$$= 2\lim_{x\to\infty}\frac{1}{\sqrt{x}} = 0$$

$\because \int_1^\infty \frac{1}{x^{3/2}}dx$ is convergent

$\therefore \int_1^\infty \frac{\log x}{x^2}dx$ is cgt

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Example

Test for Convergence the integrals

(i) $\int_0^\infty \frac{x\,\mathrm{Tan}^{-1}x}{(1+x^4)^{1/3}}$ (ii) $\int_{e^2}^\infty \frac{dx}{x\log(\log x)}$

Solutions

(i)

$$f(x) = \frac{x\,\mathrm{Tan}^{-1}x}{(1+x^4)^{1/3}} \sim \frac{x\,\mathrm{Tan}^{-1}x}{x^{4/3}}$$ Taking dominant Term

$$= \frac{\mathrm{Tan}^{-1}x}{x^{1/3}}$$ simplify

$$\sim \frac{1}{x^{1/3}}$$ $\because \mathrm{Tan}^{-1}x$ remains bounded.

Let $g(x) = \frac{1}{x^{1/3}}$

$$\frac{f(x)}{g(x)} = \frac{x^{1/3}\, Tan^{-1}x}{(1+x^4)^{1/3}} = \frac{x^{4/3}\, Tan^{-1}x}{x^{4/3}(1+x^{-4})^{1/3}}$$

$$= \frac{Tan^{-1}x}{\left(1+\frac{1}{x^4}\right)^{1/3}}$$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{Tan^{-1}(\infty)}{(1+0)^{1/3}} = \pi/2 \text{, non-zero finite}$$

$\Rightarrow$ both integrals behave alike

$\because \int_1^\infty \frac{1}{x^{1/3}}\,dx$ is dgt $\therefore \int_1^\infty f(x)\,dx$ is dgt

(ii)

$$\int_{e^2}^\infty \frac{dx}{x\log(\log x)}$$

putting $\log x = t$

$\Rightarrow x = e^t$

$dx = e^t dt = x\,dt$

<u>Limits</u> when $x = e^2$ $t = 2$

$x = \infty$ $t = \infty$

$$\int_{e^2}^\infty \frac{dx}{x\log(\log x)} = \int_2^\infty \frac{dt}{\log t}$$

Let $g(t) = \frac{1}{t^p}$

$$\lim_{t\to\infty} \frac{f(t)}{g(t)} = \lim_{t\to\infty} \frac{t^p}{\log t} = \frac{\infty}{\infty}$$

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$$= \lim_{t\to\infty} \frac{p\,t^{p-1}}{1/t} = \lim_{t\to\infty} \frac{p\,t}{t^{1-p}}$$

$$= 0 \quad \text{if } 1-p \geq 0$$
$$1 \geq p$$
$$\text{or } p \leq 1$$

$\because$ limit is zero

$\therefore$ for $p \leq 1$ $\int_2^{\infty} \frac{1}{t^p}\,dt$ will be dgt and

it does not suit

$$= \lim_{t\to\infty} \frac{p}{t^{-p}} = \infty \quad \text{if } p > 0$$

We take $p = 1$ for which $\int_2^{\infty} \frac{1}{t}\,dt$

is dgt and

$$\lim_{t\to\infty} \frac{f(t)}{g(t)} = \lim_{t\to\infty} \frac{t}{\log t} = \frac{\infty}{\infty}$$

$$= \lim_{t\to\infty} \frac{1}{1/t} = \infty$$

$\because \int_2^{\infty} \frac{1}{t}\,dt$ is dgt $\therefore \int_2^{\infty} \frac{dt}{\log t}$ is dgt

Hence $\int_{e^2}^{\infty} \frac{dx}{x \log(\log x)}$ is dgt

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Example

Check the Convergence of

$$\int_{\pi}^{\infty} \frac{1-\cos x}{x^2}\,dx$$

Solution # Let $g(x) = \frac{1}{x^p}$ $p>0$

we adjust value of p suitable for limit Comparison test as

$$f(x) = \frac{1-\cos x}{x^2}$$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1-\cos x}{x^{p+2}}$$

$$= \lim_{x\to\infty}\left(\frac{1}{x^{p+2}} - \frac{\cos x}{x^{p+2}}\right)$$

$$= 0 - 0 = 0 \quad \text{for any } p>0$$

So we can take a p +ve which makes $\int_{\pi}^{\infty} g(x)dx$ cgt. Let $p=2$

Then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty}\left(\frac{1}{x^4} - \frac{\cos x}{x^4}\right)$

$$= 0$$

$\int_{\pi}^{\infty} g(x)dx$ is Convergent

$\therefore \int_{\pi}^{\infty} \frac{1-\cos x}{x^2}\,dx$ is cgt.

Example

Test for convergence (a) $\int_{-\infty}^{-1} \frac{e^x}{x} dx$

(b) $\int_{-\infty}^{+\infty} \frac{x^3+x^2}{x^6+1} dx$

Solutions:

(a) let $x = -y$ $\quad dx = -dy$

Limits when $x = -\infty$ $\quad y = \infty$

$x = -1$ $\quad y = \infty$

$$\int_{-\infty}^{-1} \frac{e^x}{x} dx = \int_{\infty}^{1} \frac{e^{-y}}{-y}(-dy)$$

$$= -\int_{1}^{\infty} \frac{e^{-y}}{y} dy$$

Let $g(x) = \frac{1}{y^2}$

$$\lim_{y\to\infty} \frac{f(y)}{g(y)} = \lim_{y\to\infty} y e^{-y} = 0$$

$\because \int_1^{\infty} g(x)\, dx$ is cgt

$\therefore$ By L.C.T $-\int_1^{\infty} \frac{e^{-y}}{y} dy$ is cgt

$\Rightarrow \int_{-\infty}^{-1} \frac{e^x}{x} dx$ is cgt

(ii)

$$\int_{-\infty}^{+\infty} \frac{x^3+x^2}{x^6+1} dx = \underbrace{\int_{-\infty}^{0} \frac{x^3+x^2}{x^6+1} dx}_{I_1} + \underbrace{\int_{0}^{\infty} \frac{x^3+x^2}{x^6+1} dx}_{I_2}$$

In $I_1 = \int_{-\infty}^{0} \frac{x^3+x^2}{x^6+1} dx$ let $x = -y$

$$I_1 = -\int_0^{\infty} \frac{y^3-y^2}{y^6+1} dy$$

Let $g(y) = \frac{1}{y^3}$

$\lim\limits_{y\to\infty} \frac{f(y)}{g(y)} = \lim\limits_{y\to\infty} \frac{y^6-y^5}{y^6+1} = 1$, non-zero finite

$\because \int_0^{\infty} \frac{1}{y^3} dx$ is cgt $\therefore \int_0^{\infty} \frac{y^3-y^2}{y^6+1} dy$

and hence I_1 is cgt

Similarly by Taking $g(x) = \frac{1}{x^3}$, I_2 is cgt. Thus integral is cgt

---

Example

Prove that $\int_1^{\infty} \frac{\cos^2 x}{x^2} dx$ is cgt

Already solved

Example

prove that $\int_0^{\infty} \frac{\sin x}{x} dx$ Converges

Sol

$$\int_0^{\infty} \frac{\sin x}{x} dx = \int_0^{1} \frac{\sin x}{x} dx + \int_1^{\infty} \frac{\sin x}{x} dx$$

$\because \lim\limits_{x\to 0} \frac{\sin x}{x} = 1$ $\therefore$ First integral is proper. we check 2nd integral P.T.O

$$\int_x \frac{\sin x}{x}\,dx = -\frac{\cos x}{x} - \int -\cos x\left(-\frac{1}{x^2}\right)dx$$

$$= -\frac{\cos x}{x} - \int \frac{\cos x}{x^2}\,dx$$

$$\int_1^t \frac{\sin x}{x}\,dx = \cos 1 - \frac{\cos t}{t} - \int_1^t \frac{\cos x}{x^2}\,dx$$

$$\lim_{t\to\infty}\int_1^t \frac{\sin x}{x}\,dx = \cos 1 - 0 - \int_1^\infty \frac{\cos x}{x^2}\,dx$$

$$= \cos 1 - \int_1^\infty \frac{\cos x}{x^2}\,dx$$

Here $\int_1^\infty \frac{\cos x}{x^2}\,dx$ is cgt which will be proved later

Hence $\lim_{t\to\infty}\int_1^t \frac{\sin x}{x}\,dx$ is finite

$\Rightarrow \int_1^\infty \frac{\sin x}{x}\,dx$ is cgt

$\Rightarrow \int_0^\infty \frac{\sin x}{x}\,dx$ is cgt

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Example

Show that $\int_0^\infty \left(\frac{1}{x} - \frac{1}{\sinh x}\right)\frac{dx}{x}$ is cgt

Solution

$$f(x) = \left(\frac{1}{x} - \frac{1}{\sinh x}\right)\frac{1}{x}$$

$$\because \lim_{x\to 0} f(x) = \lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{\sinh x}\right)\frac{1}{x} = \frac{1}{6}$$

$\therefore$ 0 is not a point of infinite discontinuity

$$\int_0^\infty \left(\frac{1}{x} - \frac{1}{\sinh x}\right)\frac{dx}{x} = \int_0^1 \left(\frac{1}{x} - \frac{1}{\sinh x}\right)\frac{dx}{x}$$

$$+ \int_1^\infty \left(\frac{1}{x} - \frac{1}{\sinh x}\right)\frac{1}{x}\,dx$$

Convergence at $+\infty$

$$f(x) = \left(\frac{1}{x} - \frac{1}{\sinh x}\right)\frac{1}{x} = \left(\frac{1}{x} - \frac{2}{e^x - e^{-x}}\right)\frac{1}{x}$$

$$= \frac{1}{x^2} - \frac{2e^{-x}}{x(1-e^{-2x})}$$

$$= \frac{1}{x^2} - \frac{2e^{x}}{x(e^{2x}-1)} \sim \frac{1}{x^2} \text{ (dominating term at } \infty)$$

Here $\frac{1}{x^2}$ is dominating which can be checked as

$$\lim_{x\to\infty} \frac{\frac{1}{x^2}}{\frac{2e^x}{x(e^{2x}-1)}} = \lim_{x\to\infty} \frac{e^{2x}-1}{2xe^x} \qquad \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{1-\frac{1}{e^{2x}}}{2\frac{x}{e^x}}$$

$$= \frac{\lim_{x\to\infty}\left(1-\frac{1}{e^{2x}}\right)}{2\lim_{x\to\infty}\frac{x}{e^x}} = \frac{1-0}{2(0)} = \infty$$

$\Rightarrow \frac{1}{x^2}$ dominates

Note If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$, Then f leads or dominates

If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$, then g leads

Let $g(x) = \frac{1}{x^2}$

we note that

$$f(x) = \frac{1}{x^2} - \frac{2e^{-x}}{x(1-e^{-2x})} < g(x) \quad \forall x \in [1, \infty[$$

$\because \int_1^\infty g(x)\,dx$ is cgt

$\therefore \int_1^\infty f(x)\,dx$ and hence $\int_0^\infty f(x)\,dx$ is convergent.

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Example

Show that $\int_0^\infty \left(\frac{1}{1+x} - e^{-x}\right)\frac{dx}{x}$ is cgt.

<u>Sol</u> $f(x) = \left(\frac{1}{1+x} - e^{-x}\right)\frac{1}{x} = \frac{e^x - (1+x)}{x(1+x)e^x} > 0 \quad \forall x > 0$

$\because e^x > (1+x) \quad \forall x > 0$

$$\int_0^\infty \left(\frac{1}{1+x} - e^{-x}\right)\cdot\frac{1}{x}\,dx = \int_0^1 \left(\frac{1}{1+x} - e^{-x}\right)\frac{dx}{x} + \int_1^\infty \left(\frac{1}{1+x} - e^{-x}\right)\cdot\frac{1}{x}\,dx$$

$$\because \lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{e^x - 1 - x}{xe^x + x^2e^x} \quad \frac{0}{0}$$

$$= \cancel{\lim_{x\to\infty} \frac{\frac{1}{x^2} - \frac{1}{x^2e^x} - \frac{1}{xe^x}}{\frac{1}{x} + 1}} = \frac{0}{0+1} = 0$$

$$= \lim_{x\to 0} \frac{e^x - 1}{xe^x + e^x + x^2e^x + 2xe^x} = \frac{0}{1} = 0$$

$\Rightarrow 0$ is not point of infinite discontinuity.

For $\int_1^\infty \left(\frac{1}{1+x} - e^{-x}\right)\frac{1}{x}\,dx$

$$f(x) = \left(\frac{1}{1+x} - e^{-x}\right)\frac{1}{x}$$

$$= \frac{1}{x+x^2} - \frac{e^{-x}}{x}$$

$$= \frac{1}{x+x^2} - \frac{1}{xe^x}$$

$$= \frac{e^x - (1+x)}{(x+x^2)xe^x}$$

$$= \frac{e^x - (1+x)}{x^2e^x + xe^x}$$

$$\sim \frac{e^x}{x^2e^x} = \frac{1}{x^2}$$ [Taking dominating Terms]

Let $g(x) = \frac{1}{x^2}$

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{e^x-(1+x)}{e^x}\cdot\frac{x}{1+x}$$

$$= \lim_{x\to\infty}\left(\frac{e^x-(1+x)}{e^x}\right)\cdot\lim_{x\to\infty}\frac{x}{1+x}$$

$$= \left(1 - \lim_{x\to\infty}\frac{1+x}{e^x}\right)\times 1 = (1-0)\times 1 = 1$$

$\because \int_1^\infty g(x)\,dx$ is cgt $\therefore I_2 = \int_1^\infty f(x)\,dx$ & hence

$\int_0^\infty \left(\frac{1}{1+x} - e^{-x}\right)\frac{dx}{x}$ is cgt

Example

Test for convergence $\int_0^\infty \frac{x^{2m}}{1+x^{2n}}dx$ where m, n being +ve integer

Sol

$$\int_0^\infty \frac{x^{2m}}{1+x^{2n}}dx = \int_0^1 \frac{x^{2m}}{1+x^{2n}}dx + \int_1^\infty \frac{x^{2m}}{1+x^{2n}}dx$$

$$= I_1 + I_2$$

I_1 is proper integral

For $I_2 = \int_1^\infty \frac{x^{2m}\,dx}{1+x^{2n}}$

$f(x) = \frac{x^{2m}}{1+x^{2n}}$ Let $g(x) = \frac{1}{x^{2n-2m}}$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^{2m}}{1+x^{2n}} \times x^{2n-2m}$$

$$= \lim_{x\to\infty} \frac{x^{2m}}{1+x^{2n}} \times \frac{x^{2n}}{x^{2m}}$$

$$= \lim_{x\to\infty} \frac{x^{2n}}{1+x^{2n}} = 1, \text{ finite non-zero}$$

$\Rightarrow$ both integrals $\int_1^\infty f(x)dx$ & $\int_1^\infty g(x)dx$ behave alike.

So I_2 is cgt if $2n-2m > 1$, which is possible if $n > m$ and I_2 is dgt if $2n-2m \leq 1$ which is possible if $n \leq m$. Thus $\int_0^\infty \frac{x^{2m}}{1+x^{2n}}$ is cgt if $n > m$ and is dgt if $n \leq m$

Example

Show That the improper integral $\int_0^\infty \log(1+2\operatorname{sech} x)\,dx$ Converge

Sol

$\because \log(1+x) < x \quad \forall x > 0$

$\therefore \log(1+2\operatorname{sech}x) < 2\operatorname{sech}x = \frac{4}{e^x+e^{-x}} < \frac{4}{e^x} = 4e^{-x}$

$\because \int_0^\infty 4e^{-x}dx$ is cgt

$\therefore \int_0^\infty \log(1+2\operatorname{sech}x)dx$ is cgt.

Example

(a) Show that $\int_0^\infty \frac{\cosh bt}{\cosh at}dt \quad a>0, b>0$ Converge iff $b<a$

(b) Show that $\int_0^\infty \frac{\sinh bx}{\sinh ax}dx \quad a>0, b>0$ Converge iff $a>b$

Solutions

(a) $f(x) = \frac{\cosh bt}{\cosh at}$

Case I when $b<a$

$\Rightarrow \frac{\cosh bt}{\cosh at} = \frac{e^{bt}+e^{-bt}}{e^{at}+e^{-at}} < \frac{e^{bt}+e^{bt}}{e^{at}}$

$= \frac{2e^{bt}}{e^{at}} = 2e^{-(a-b)t}$

Now $2\int_0^\infty e^{-(a-b)t}dx$ is cgt by $\int_0^\infty e^{-\lambda x}dx$ is cgt iff $\lambda>0$

$\Rightarrow \int_0^\infty f(x)\,dx$ is cgt

Case II when $b > a$

$$f(x) = \frac{\cosh bt}{\cosh at} = \frac{e^{bt} + e^{-bt}}{e^{at} + e^{-at}} > \frac{e^{bt}}{e^{at} + e^{at}}$$

$$= \frac{1}{2} e^{(b-a)t} = \frac{1}{2} e^{-(a-b)t}$$

Here $a - b < 0$

Now $\frac{1}{2}\int_0^\infty e^{-(a-b)t}\,dt$ is dgt by $\int_0^\infty e^{-\lambda x}\,dx$ is dgt if $\lambda \leq 0$

By Comparison test $\int_0^\infty f(t)\,dt$ is dgt

(b)

Case I when $a > b$

$$\frac{\sinh bx}{\sinh ax} = \frac{e^{bx} - e^{-bx}}{e^{ax} - e^{-ax}} < \frac{e^{bx}}{e^{ax} - 1}$$

$\because e^{-ax} < 1$

We check convergence of $\int_0^\infty \frac{e^{bx}}{e^{ax} - 1}\,dx$

$$f(x) = \frac{e^{bx}}{e^{ax} - 1} \sim \frac{e^{bx}}{e^{ax}} \quad \because a > 0,\ b > 0$$

$$= e^{(b-a)x}$$

$$= e^{-(a-b)x}$$

let $g(x) = \frac{e^{bx}}{e^{ax}} = e^{-(a-b)x}$ $\quad a - b > 0$

Then $\int_0^\infty g(x)\,dx$ is cgt

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{e^{bx}}{e^{ax} - 1} \times \frac{e^{ax}}{e^{bx}} = 1$$

$\Rightarrow$ both integrals behave alike

$\Rightarrow \int_0^\infty \frac{e^{bx}}{e^{ax}-1}\,dx$ is cgt

Hence by Comparison test $\int_0^\infty \frac{\sinh bx}{\sinh ax}\,dx$ is cgt

<u>Case II</u> $a < b \Rightarrow$

$$\frac{\sinh bx}{\sinh ax} = \frac{e^{bx}-e^{-bx}}{e^{ax}-e^{-ax}} > \frac{e^{bx}-1}{e^{ax}}$$

$\because e^{-bx} < 1$ $\quad b>0,\ x>0$

$\Rightarrow -e^{-bx} > -1$

$\Rightarrow e^{bx}-e^{-bx} > e^{bx}-1$

& $e^{ax}-e^{-ax} < e^{ax}$

Now we check $\int_0^\infty \frac{e^{bx}-1}{e^{ax}}\,dx$ for convergence

$$f(x) = \frac{e^{bx}-1}{e^{ax}} \qquad \text{let } g(x) = \frac{e^{bx}}{e^{ax}}$$

$$= e^{(b-a)x}$$

$$= e^{-(a-b)x} \qquad a-b<0$$

Then $\int_0^\infty g(x)\,dx$ is dgt

and $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{e^{bx}-1}{e^{bx}} = 1$

Thus $\int_0^\infty \frac{e^{bx}-1}{e^{ax}}$ is dgt and by Comparison test $\int_0^\infty \frac{\sinh bx}{\sinh ax}$ is dgt

OR

$$f(x) = \frac{e^{bx} - e^{-bx}}{e^{ax} - e^{-ax}} \sim \frac{e^{bx}}{e^{ax}} = e^{-(a-b)x}$$

Let $g(x) = e^{-(a-b)x}$. Then $\int_0^\infty e^{-(a-b)x}\,dx$

is dgt because $a-b<0$

Now $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = \lim\limits_{x\to\infty} \frac{e^{bx} - e^{-bx}}{e^{ax} - e^{-ax}} \times \frac{e^{ax}}{e^{bx}}$

$$= \lim_{x\to\infty} \frac{1 - e^{-2bx}}{1 - e^{-2ax}} = \lim_{x\to\infty} \frac{1 - \frac{1}{e^{2bx}}}{1 - \frac{1}{e^{2ax}}}$$

$$= 1$$

Thus by Limit Comparison test

$\int_0^\infty \frac{\sinh bx}{\sinh ax}\,dx$ is dgt

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Example

$$\int_1^\infty \frac{\sin^2(\frac{1}{x})}{\sqrt{x}}\,dx$$

Sol $\because \sin x \simeq x$ when $x \to 0$

and $\frac{1}{x} \to 0$ as $x \to \infty$

$\therefore$ as $x \to \infty$ $\sin(\frac{1}{x}) \simeq 1/x$

$$\Rightarrow \frac{\sin^2(1/x)}{\sqrt{x}} \simeq \frac{(1/x)^2}{\sqrt{x}} = \frac{1}{x^2\sqrt{x}} = \frac{1}{x^{5/2}}$$

Let $g(x) = \frac{1}{x^{5/2}}$. Then $\int_1^\infty g(x)\,dx$ is cgt

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} x^2 \sin(1/x) = \lim_{x\to\infty} \left[\frac{\sin(1/x)}{1/x}\right]^2$$

$= \lim_{y \to 0} \left(\frac{\sin y}{y}\right)^2 = 1$ putting $\frac{1}{x} = y$

$\Rightarrow$ By L.C.T $\int_1^\infty \frac{\sin^2(\frac{1}{x})}{\sqrt{x}} dx$ is cgt

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Example

Show that $\int_a^\infty \frac{\sin^2 x}{x^p} dx$ $(a > 0,\ p > 1)$

is cgt

Solution

$\because\ 0 \leq \frac{\sin^2 x}{x^p} \leq \frac{1}{x^p}$ $\forall x \in [a, \infty[$, $a > 0$

and $\int_a^\infty \frac{1}{x^p} dx$ is cgt for $p > 1$

$\therefore$ By Comparison test $\int_a^\infty \frac{\sin^2 x}{x^p} dx$

is cgt when $a > 0,\ p > 1$

Example

Show that $\int_1^\infty \frac{\log x}{x^p} dx$ is cgt if $p > 1$

and is dgt. if $p \leq 1$

Solution

For $\lambda > 0$

$\lim_{x \to \infty} x^{p-\lambda} f(x) = \lim_{x \to \infty} x^{p-\lambda} \frac{\log x}{x^p}$

$= \lim_{x \to \infty} \frac{\log x}{x^\lambda} = 0$, finite

Hence $\int_1^\infty f(x)\,dx$ is cgt

if $P-\lambda > 1$

if $P > 1+\lambda$ for $\lambda > 0$

i.e $P > 1$

Again let $g(x) = \frac{1}{x^P}$

$$\lim_{x\to\infty} \frac{f}{g} = \lim_{x\to\infty} \log x = \infty$$

and $\int_1^\infty \frac{1}{x^P}\,dx$ is dgt if $P \le 1$

$\Rightarrow \int_1^\infty f(x)\,dx$ is dgt if $P \le 1$

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Example

Examine the convergence of

$$\int_1^\infty \left[1 - \cos(2/x)\right] dx$$

Solution

Let $f(x) = 1 - \cos(2/x)$ $x \geq 1$

and $g(x) = \frac{2}{x^2}$

Then $\int_1^\infty \frac{2}{x^2}\,dx$ is cgt

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{2\sin^2(1/x)}{2/x^2}$$

$$= \lim_{x\to\infty} \left[\frac{\sin(1/x)}{1/x}\right]^2$$

$$= \lim_{y\to 0} \left(\frac{\sin y}{y}\right)^2 = 1$$ putting $1/x = y$

P.T.O

Example

Show that $\int_1^\infty x^k\left(\frac{x+\sin x}{x-\sin x}\right)dx$

is cgt iff $k<-1$

Solution

Let $f(x) = x^k\left(\frac{x+\sin x}{x-\sin x}\right)$

and $g(x) = x^k$

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{x+\sin x}{x-\sin x}$$

$$= \lim_{x\to\infty}\frac{1+\frac{\sin x}{x}}{1-\frac{\sin x}{x}} = \frac{1-0}{1+0}$$

($\because$ Sin x is bounded & $\frac{1}{x}\to 0$ as $x\to\infty$
$\therefore \lim_{x\to\infty}\frac{1}{x}\sin x = 0$)

Now $\int_1^\infty x^k\,dx = \int_1^\infty \frac{1}{x^{-k}}\,dx$ is

cgt iff $-k>1$ i.e. iff $k<-1$

thus $\int_1^\infty f(x)\,dx$ is cgt iff $k<-1$

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Example

Show that $\int_2^\infty \frac{dx}{x^k \log x}$ converges for $k>1$ and diverges for $k \le 1$

Solution

Let $f(x) = \frac{1}{x^k \log x}$ $\quad g(x) = \frac{1}{x^k}$

$x \geq 2$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1}{\log x} = 0,$$

Now $\int_2^\infty \frac{1}{x^k}$ is cgt if $k>1$

By Comparison test $\int_2^\infty \frac{dx}{x^k \log x}$ is cgt for $k>1$

For $k=1$

$$\int_2^\infty \frac{1}{x\log x}\,dx \qquad f(x) = \frac{1}{x\log x}$$

$$\lim_{t\to\infty}\int_2^t \frac{1}{x\log x}\,dx = \lim_{t\to\infty}\left[\log(\log x)\right]_2^t$$

$$= \lim_{t\to\infty}\left[\log(\log t) - \log(\log 2)\right]$$

$$= \infty$$

$\Rightarrow$ Integral diverges for $k=1$

For $k<1$

For $k<1$, $x^k < x \quad \forall x \geq 2$

$$\Rightarrow x^k \log x < x \log x \quad \forall x \geq 2$$

$$\Rightarrow \frac{1}{x^k \log x} > \frac{1}{x \log x} \quad \forall x \geq 2$$

$\because \int_2^\infty \frac{1}{x \log x} dx$ is dgt

$\therefore \int_2^\infty \frac{1}{x^k \log x} dx$ is also dgt

---

General Test for Convergence at ∞

(Integrand may change sign)

Cauchy's Test

Assume that $f \in R[a,t]$ $\forall t \geq a$. Then the integral $\int_a^\infty f dx$ converges iff for every $\epsilon > 0$ there exists a number k such that

$$\left| \int_{t_1}^{t_2} f(x) dx \right| < \epsilon \quad \forall t_1, t_2 > k$$

Proof # Let $F(t) = \int_a^t f(x) dx \quad \forall t \geq a$

Let $\int_a^\infty f(x) dx$ be convergent. Then $\lim_{t \to \infty} F(t)$ exists. Let $\lim_{t \to \infty} F(t) = A$

By definition of limit of a function at ∞, for every $\epsilon > 0$, $\exists$ a no k such that

$$|F(t_1) - A| < \epsilon/2 \quad \forall\, t_1 > k$$

$$\Rightarrow \left| \int_a^{t_1} f(x)\,dx - A \right| < \epsilon/2 \quad \forall\, t_1 > k. \to ①$$

Also, for $t_2 > t_1 > k$

$$|F(t_2) - A| < \epsilon/2$$

$$\Rightarrow \left| \int_a^{t_2} f(x)\,dx - A \right| < \epsilon/2 \quad \to ②$$

Now

$$\left| \int_{t_1}^{t_2} f(x)\,dx \right| = \left| \int_a^{t_2} f(x)\,dx - \int_a^{t_1} f(x)\,dx \right|$$

$$= \left| \int_a^{t_2} f(x)\,dx - A + A - \int_a^{t_1} f(x)\,dx \right|$$

$$\leq \left| \int_a^{t_2} f(x)\,dx - A \right| + \left| A - \int_a^{t_1} f(x)\,dx \right|$$

$$< \epsilon/2 + \epsilon/2 = \epsilon \quad \text{when } t_2 > t_1 > k$$

$$\Rightarrow \left| \int_{t_1}^{t_2} f(x)\,dx \right| < \epsilon \quad \forall\, t_2 > t_1 > k$$

<u>Converse</u>

Suppose that Cauchy condition holds.

For any +ve integer $n \geq a$ define

Consider n, m such that $n > m > k$

Then Cauchy condition holds for n, m & we have

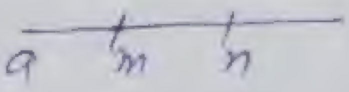
$$\left|\int_m^n f\,dx\right| < \epsilon$$

Now

$$|a_n - a_m| = \left|\int_a^n f(x)\,dx - \int_a^m f(x)\,dx\right|$$

$$\Downarrow$$

$$= \left|\left(\cancel{\int_a^m f(x)\,dx} + \int_m^n f(x)\,dx\right) - \cancel{\int_a^m f(x)\,dx}\right|$$



$$= \left|\int_m^n f(x)\,dx\right| < \epsilon$$

Thus for any $\epsilon > 0$, we have traced a no k such that

$$|a_n - a_m| < \epsilon \qquad \forall\ n > m > k$$

$\Rightarrow \{a_n\}$ is a Cauchy sequence of real numbers

$\Rightarrow \{a_n\}$ is convergent

Let $\lim_{n \to \infty} a_n = l$

Then for, given $\epsilon > 0$, $\exists$ a no k_0 such that

$$|a_n - l| < \epsilon/2 \qquad \forall\ n > k_0 \rightarrow (3)$$

Now if $k_0 < k$ (above in cauchy condition), then condition (3) is also true $\forall\ n > k > k_0$ and if $k_0 > k$, then again (3) is true $\forall\ n > k > k_0$. So we can adjust (3)

for k ℓ

$$|a_n - \ell| < \epsilon/2 \quad \forall n > k \to (4)$$

Also by Cauchy's condition

$$\left|\int_{t_1}^{t_2} f(x)\,dx\right| < \epsilon/2 \quad t_2 > t_1 > k \to (5)$$

Now if $n, a, t_1 > k$ and $t_2 \geq k+1$, we have

$$\left|\int_a^{t_1} f(x)\,dx - A\right| = \left|\int_a^n f\,dx - \ell + \int_n^{t_1} f(x)\,dx\right|$$

$$\leq |a_n - \ell| + \left|\int_n^{t_1} f(x)\,dx\right|$$

$$< \epsilon/2 + \epsilon/2 = \epsilon \quad \forall t_1 \geq a$$

$t_1 > k$

$$\Rightarrow \lim_{t_1 \to \infty} \int_a^{t_1} f(x)\,dx = \ell$$

$$\Rightarrow \int_a^{\infty} f(x)\,dx \text{ is cgt.}$$

OR

Let $F(t) = \int_a^t f(x)\,dx$

Now $\lim_{t \to \infty} F(t)$ will exist

iff for every $\epsilon > 0$ $\exists$ a no k such that

$$|F(t_2) - F(t_1)| < \epsilon \quad \forall t_1, t_2 > k$$

$$\text{iff} \quad \left|\int_a^{t_2} f(x)\,dx - \int_a^{t_1} f(x)\,dx\right| < \epsilon \quad \text{" "}$$

$$\text{iff} \quad \left|\int_{t_1}^{t_2} f(x)\,dx\right| < \epsilon \quad \forall t_2 > t_1 > k$$

proved.

> Rieview

- If $\lim_{x\to\infty} f(x) = l$, then for any $\epsilon > 0$, $\exists$ a no $k > 0$ such that $|f(x) - l| < \epsilon \quad \forall x > k$.
- A sequence is said to be cgt if $\exists$ a no l such that for every $\epsilon > 0$, there exists a +ve integer n_0 (n_0 may be +ve real number) such that

$$|a_n - l| < \epsilon \qquad \forall n \geq n_0$$

$$|a_n - l| < \epsilon \qquad \forall n > n_0 \text{ (when } n_0 \text{ is not integer)}$$

- A sequence $\{a_n\}$ is said to be cauchy if for every $\epsilon > 0$, $\exists$ a +ve integer n_0 such that

$$|a_n - a_m| < \epsilon \qquad \forall n, m \geq n_0$$

- A sequence of real number is cauchy iff it is cgt

Example

(a) Use cauchy criterion to prove that $\int_1^\infty \frac{\sin x}{x} dx$ is cgt

(b) Show that $\int_0^\infty \frac{\sin x}{x} dx$ is cgt

Solution

Let $\epsilon > 0$ be given. Let $t_1, t_2 \in [1, \infty[$

$$\int_{t_1}^{t_2} \frac{\sin x}{x} dx = \left[-\frac{\cos x}{x}\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{\cos x}{x^2} dx$$

$$\left|\int_{t_1}^{t_2} \frac{\sin x}{x} dx\right| = \left|\frac{\cos t_1}{t_1} - \frac{\cos t_2}{t_2} - \int_{t_1}^{t_2} \frac{\cos x}{x} dx\right|$$

$$\leq \left|\frac{\cos t_1}{t_1}\right| + \left|\frac{\cos t_2}{t_2}\right| + \left|\int_{t_1}^{t_2} \frac{\cos x}{x} dx\right|$$

$$\leq \left|\frac{\cos t_1}{t_1}\right| + \left|\frac{\cos t_2}{t_2}\right| + \int_{t_1}^{t_2} \left|\frac{\cos x}{x^2}\right| dx$$

$\because \left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$

$$\leq \frac{1}{t_1} + \frac{1}{t_2} + \int_{t_1}^{t_2} \frac{1}{x^2} dx \qquad \because |\cos x| \leq 1$$

$$= \frac{1}{t_1} + \frac{1}{t_2} - \left[\frac{1}{x}\right]_{t_1}^{t_2} = \frac{1}{t_1} + \cancel{\frac{1}{t_2}} + \frac{1}{t_1} - \cancel{\frac{1}{t_2}}$$

$$= \frac{2}{t_1}$$

$$\left|\int_{t_1}^{t_2} f(x)dx\right| < \frac{2}{t_1} \longrightarrow (1)$$

Let $\frac{2}{t_1} < \epsilon$, then $t_1 > 2/\epsilon = k$

and $t_2 > t_1 > k$. Thus for any $\epsilon > 0$, we can find a no $k = 2/\epsilon$ such that

$$\left|\int_{t_1}^{t_2} f(x)dx\right| < \epsilon \qquad \forall\ t_1, t_2 > k$$

$\Rightarrow \int_1^\infty \frac{\sin x}{x} dx$ is cgt

(b)

$$\int_0^\infty \frac{\sin x}{x} dx = \int_0^1 \frac{\sin x}{x} dx + \int_1^\infty \frac{\sin x}{x} dx$$

$\because \lim_{x \to 0} \frac{\sin x}{x} = 1$

$\therefore$ 0 is not point of infinite discontinuity and 1st integral is proper

Also $\int_1^\infty \frac{\sin x}{x} dx$ is cgt as proved in (a)

$\Rightarrow \int_0^\infty \frac{\sin x}{x} dx$ is cgt

Absolute Convergence

The improper integral $\int_a^\infty f(x)\,dx$ is said to be absolutly cgt. if $\int_a^\infty |f(x)|dx$ is cgt

Conditionally Convergent

A cgt integral which is not absolutly convergent. i.e if $\int_a^\infty f(x)dx$ is cgt but $\int_a^\infty |f(x)|dx$ is dgt.

Theorem# Every absolutly convergent integral is convergent i.e

$$\int_a^\infty |f(x)|dx \text{ Converges} \Rightarrow \int_a^\infty f(x)dx \text{ Converges}$$

Proof# $\because \int_a^\infty |f(x)|dx$ converges

$\therefore$ For every $\epsilon > 0$ $\exists$ a +ve no k such that

$$\left|\int_{t_1}^{t_2} |f(x)|dx\right| < \epsilon \quad \forall\, t_1, t_2 > k$$

Also $\left|\int_{t_1}^{t_2} f(x)dx\right| \leq \left|\int_{t_1}^{t_2} |f(x)|dx\right| < \epsilon \quad \forall\, t_1, t_2 > k$ $\quad \because f(x) \leq |f(x)|$

By Cauchy criterion. $\int_a^\infty f(x)dx$ is cgt

Note Converse of the above theorem may not be true e.g $\int_1^\infty \frac{\sin x}{x}dx$ is cgt but $\int_1^\infty \left|\frac{\sin x}{x}\right|dx$ is dgt. which will be proved later

Improper Integral of 1st kind and infinite Series

(1) **Theorem#** Every Convergent infinite integral $\int_a^\infty f(x)\,dx$ can be written as a Convergent infinite Series In fact we have

$$\int_a^\infty f(x)\,dx = \sum_{k=1}^\infty a_k \quad \text{where } a_k = \int_{a+k-1}^{a+k} f(x)\,dx$$

Proof# let $a_n = \int_{a+n-1}^{a+n} f(x)\,dx$

Then
$$\int_a^{a+n} f(x)\,dx = \int_a^{a+1} + \int_{a+1}^{a+2} + \int_{a+2}^{a+3} + \cdots + \int_{a+n-1}^{a+n}$$

$$= a_1 + a_2 + \cdots + a_n$$

$$= \sum_{k=1}^{n} a_k$$

$\because \int_a^\infty f(x)\,dx$ is cgt

$$\therefore \lim_{n\to\infty} \int_a^{a+n} f(x)\,dx = \text{finite} = \int_a^\infty f(x)\,dx$$

$$\Rightarrow \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} \int_a^{a+n} = \int_a^\infty f(x)\,dx$$

$$\Rightarrow \sum_{k=1}^{\infty} a_k = \int_a^\infty f(x)\,dx$$

Thus series is also cgt

P.T.O

Remarks# note that Convergence of the related series does not always imply the Convergence of the integral e.g.

Consider integral $\int_0^\infty \sin 2\pi x\, dx$.

Let $a_k = \int_{k-1}^{k} \sin 2\pi x\, dx$. Then

$a_k = 0 \quad \forall k$

and $\sum a_k$ converges.

But $\int_0^\infty \sin 2\pi x\, dx = \lim_{t\to\infty} \int_0^t \sin 2\pi x\, dx$

$= \lim_{t\to\infty} \frac{1-\cos 2\pi t}{2\pi}$ does not exis

$\Rightarrow \int_0^\infty \sin 2\pi x\, dx$ diverges

But we have following theorem

(2) **Cauchy - Maclaurin Integral Test**

Let f be +ve, decreasing function with domain $[1\ \infty[$. Then the series $\sum_{n=1}^{\infty} f(n)$ is cgt. iff the integral $\int_{a=1}^{\infty} f(x)\, dx$ is cgt.

Proof# $\because$ f is monotone decreasing $\forall t \geq 1$.

$\therefore$ f is integrable on $[1\ t]$

Let $N \geq 1$ be an integer. For all integer k such that $1 \leq k \leq N+1$, we have

$f(k+1) \leq f(x) \leq f(k) \quad \forall x \in [k, k+1]$

$\Rightarrow f(k+1) \leq \int_k^{k+1} f(x) \leq f(k)$

$\Rightarrow f(2)+f(3)+\cdots+f(N)\le\int_1^N f(x)dx\le f(1)+f(2)+\cdots f($ $\longrightarrow$ ①

If $\sum_{n=1}^{\infty} f(n)$ is cgt with sum S (say)

, then $f(1)+f(2)+\cdots f(N-1)\le S$

and from ①

$$\int_1^N f(x)dx \le S \qquad \forall N>1$$

$\Rightarrow \int_1^{\infty} f(x)dx$ is cgt

If $\sum_{n=1}^{\infty} f(n)$ is dgt, then $f(2)+f(3)+\cdots+f$ $\to\infty$ as $N\to\infty$ and $\int_1^N f(x)dx\to\infty$ as $N\to\infty$

$\Rightarrow \int_1^{\infty} f(x)dx$ is dgt

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Example

Show that $\int_1^{\infty}\frac{\sin x}{x^p}dx$ Converges absolutely if $p>1$

Solution

$$f(x)=\frac{\sin x}{x^p} \qquad x\ge 1$$

$$|f(x)|=\frac{|\sin x|}{x^p}\le\frac{1}{x^p} \qquad \because |\sin x|\le 1 \quad \forall x\ge 1$$

$\because \int_1^{\infty}\frac{1}{x^p}dx$ is cgt if $p>1$

$\therefore \int_1^{\infty}|f(x)|dx$ is cgt if $p>1$

Example

Discuss the Convergence of integral $\int_1^\infty f(x)\,dx$, where

$$f(x) = \begin{cases} \frac{1}{x^2} & x \text{ is rational} \\ -\frac{1}{x^2} & x \text{ is irrational} \end{cases}$$

Solution

$$|f(x)| = \frac{1}{x^2}$$

$\because \int_1^\infty |f(x)|\,dx = \int_1^\infty \frac{1}{x^2}\,dx$ is cgt

$\therefore \int_1^\infty f(x)\,dx$ is also cgt.

Example

Show that the integral $\int_1^\infty \frac{\cos x}{\sqrt{1+x^3}}\,dx$ is absolutely cgt.

Solution

$$f(x) = \frac{\cos x}{\sqrt{1+x^3}} \qquad |f(x)| = \frac{|\cos x|}{\sqrt{1+x^3}}$$

$$\leq \frac{1}{\sqrt{1+x^3}} \leq \frac{1}{\sqrt{x^3}} = \frac{1}{x^{3/2}}$$

$\forall x \geq 1$

$\because \int_1^\infty \frac{1}{x^{3/2}}\,dx$ is cgt

$\therefore \int_1^\infty |f(x)|\,dx$ & hence $\int_1^\infty f(x)\,dx$ is cgt

So given integral is absolutely cgt

> **Concept** If f is a periodic function of period T. then
$$\int_{mT}^{nT} f(x)\,dx = (n-m)\int_0^T f(x)\,dx$$

Example

Show that $\int_0^\infty \frac{\sin x}{x}dx$ is cgt but $\int_0^\infty \frac{|\sin x|}{x}dx$ is not cgt

Solution

$$\int_0^\infty \frac{\sin x}{x}dx = \underbrace{\int_0^1 \frac{\sin x}{x}dx}_{I_1} + \underbrace{\int_1^\infty \frac{\sin x}{x}dx}_{I_2}$$

$\because \lim_{x\to 0^+} \frac{\sin x}{x} = 1 \quad \therefore I_1$ is proper

I_2 has already been proved

$I_2 = \int_1^\infty \frac{\sin x}{x}dx$ (II Method)

$f(x) = \frac{\sin x}{x}$ is bound and integrable on $[t_1, t_2]$ where $t_2 > t_1 > 1$. $\frac{1}{x}$ is monotone decreasing in $[t_1, t_2]$

By 2nd Mean value theorem (Weierstrass form)

$\exists$ a pt $t_0 \in [t_1, t_2]$ such that

$$\int_{t_1}^{t_2} \frac{\sin x\,dx}{x} = \frac{1}{t_1}\int_{t_1}^{t_0} \sin x\,dx + \frac{1}{t_2}\int_{t_0}^{t_2} \sin x\,dx$$

$$= \frac{1}{t_1}(\cos t_1 - \cos t_0) + \frac{1}{t_2}(\cos t_0 - \cos t_2)$$

$$\left|\int_{t_1}^{t_2} \frac{\sin x}{x}dx\right| \le \frac{2}{t_1} + \frac{2}{t_2} < \frac{4}{t_1} \quad (\because t_2 > t_1)$$

Let $\frac{4}{t_1} < \epsilon \Rightarrow t_1 > \frac{4}{\epsilon} = k$

Also $\frac{4}{t_2} < \frac{4}{t_1} < \epsilon \quad \therefore t_2 > t_1 > k$

Thus for any $\epsilon > 0$, we can find $k = \frac{4}{\epsilon}$ such that

$$\left| \int_{t_1}^{t_2} \frac{\sin x}{x} dx \right| < \epsilon \qquad \forall\, t_1, t_2 > k$$

Thus $\int_1^\infty \frac{\sin x}{x} dx$ is cgt. Hence $\int_0^\infty \frac{\sin x}{x} dx$ Convergent

Next we show that $\int_0^\infty \frac{|\sin x|}{x} dx$ is not cgt

For $n \in N$

$$\int_0^{n\pi} \frac{|\sin x|}{x} dx = \sum_{i=1}^{n} \int_{(i-1)\pi}^{i\pi} \frac{|\sin x|}{x} dx$$

$$\geq \sum_{i=1}^{n} \frac{1}{i\pi} \int_{(i-1)\pi}^{i\pi} |\sin x| dx$$

on $[(i-1)\pi, i\pi]$ $\frac{|\sin x|}{x} \geq \frac{|\sin x|}{i\pi}$ $\because x \leq i\pi$

$$= \frac{1}{\pi} \sum_{i=1}^{n} \frac{1}{i} \frac{\cancel{n}}{\cancel{\pi}} [\cancel{\pi} - (i-1)\cancel{\pi}] \pi \int_0^{\pi} |\sin x| dx$$

$\because |\sin x|$ is periodic function with period π

$$= \frac{1}{\pi} \sum_{i=1}^{n} \frac{[i-(i-1)]\pi}{i} \int_0^{\pi} |\sin x| dx$$

$$= \frac{1}{\pi} \sum_{i=1}^{n} \frac{\pi}{i} \int_0^{\pi} |\sin x| dx$$

$\sin x \geq 0$ on $[0, \pi]$

$$= \sum_{i=1}^{n} \frac{1}{i} \int_0^{\pi} \sin x \, dx$$

$$= [-\cos x]_0^{\pi} \sum_{i=1}^{n} \frac{1}{i}$$

$$= 2 \sum_{i=1}^{n} \frac{1}{i}$$

$$\int_0^{n\pi} \frac{|\sin x|}{x} dx \geq 2 \sum_{i=1}^{n} \frac{1}{i} = 2 \sum_{n=1}^{n} \frac{1}{n}$$

Now when $n \to \infty$ series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to to $+\infty$ So $\lim_{n \to \infty} \int_0^{n\pi} \frac{|\sin x|}{x} dx$ diverges to $+\infty$

For any real no k ($k > 0$), $\exists$ always exists a natural no n such that

$$n\pi \leq k < (n+1)\pi$$

$$\int_0^{k} \frac{|\sin x|}{x} dx \geq \int_0^{n\pi} \frac{|\sin x|}{x} dx$$

as $k \to \infty$, n also tends to $+\infty$, hence $\int_0^{\infty} \frac{|\sin x|}{x}$ diverges to $+\infty$. Thus $\int_0^{\infty} \frac{|\sin x|}{x} dx$ is dgt

So $\int_0^{\infty} \frac{\sin x}{x} dx$ is Conditionally convergent

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Test for absolute Convergene when integrand is Product of Two Function

Theorem# Let $f : [a, \infty) \to \mathbb{R}$ be bounded on $[a, \infty)$, integrable on every closed sub-interval of $[a, \infty)$ i.e $f \in R[a, t]$ $\forall t \geq a$ and $\int_a^{\infty} g\, dx$ is absolutely cgt at ∞. Then $\int_a^{\infty} f(x) g(x) dx$ is absolutely cgt

Proof# Since $f(x)$ is bounded for all $x \geq a$, $\exists$ a no $K>0$ such that

$$|f(x)| \leq K \quad \forall x \geq a \rightarrow (1)$$

Since $\int_a^\infty g(x)\,dx$ is absolutely Convergent i.e $\int_a^\infty |g(x)|\,dx$ is cgt, therefore $\exists$ $M>0$ such that

$$\int_a^t |g(x)|\,dx \leq M \quad \forall t \geq a \rightarrow (2)$$

$$\int_a^t |f(x)\,g(x)|\,dx = \int_a^t |f(x)|\,|g(x)|\,dx$$

$$\leq K \int_a^t |g(x)|\,dx$$

$$\leq KM \quad \forall t \geq a$$

Hence $\int_a^\infty |f(x)\,g(x)|\,dx$ is cgt.

$\Rightarrow$ $\int_a^\infty f(x)\,g(x)\,dx$ is absolutely cgt.

Review#

- Second Mean Value Theorem (Bonnet's theorem)
  f, g's $\in R[a,b]$, then there exist point $c \in [a,b]$ such that
  $$\int_a f(x)\,g(x)\,dx = f(a)\int_a^c g(x)\,dx + f(b)\int_c^b g(x)\,dx$$
-

Test for infinite integral of Product of functions

Abel's Test # If $f(x)$ is bounded and monotone for all $x \geq a$ and $\int_a^\infty g(x)dx$ is cgt, then $\int_a^\infty f(x)g(x)\,dx$ is cgt

OR

If $\int_a^\infty g(x)dx$ is cgt, then by insertion of a bounded monotone function f on $[a\ \infty)$, then $\int_a f(x)g(x)dx$ is cgt i.e convergence is not affected by insertion of bounded monotone function

Proof # The function f, which is monotone in $[a\ \infty)$ and monotone in $[a\ \infty)$, is integrable in $[a\ t]$ $t \geq a$.

Applying the 2nd mean value theorem we have a point $c \in [t_1\ t_2]$ s.t

$$\int_{t_1}^{t_2} g(x)f(x)\,dx = f(t_1)\int_{t_1}^{c} g(x)\,dx + f(t_2)\int_{c}^{t_2} g(x)\,dx \to (1)$$

Since f is given to be bounded on $[a\ \infty)$, there exists +ve no K such that,

$$|f(x)| \leq K \quad \forall x \geq a$$

In particular

$$|f(t_1)| \leq k \quad \& \quad |f(t_2)| \leq k \to (2)$$

Also $\int_a^\infty g(x)\,dx$ is cgt, by Cauchy criterion for every $\epsilon > 0$, there exists +ve no N s.that

$$\left|\int_{t_1}^{t_2} g(x)\,dx\right| < \frac{\epsilon}{2k} \quad \forall\, t_1, t_2 \geq N \rightarrow ③$$

Let the nos t_1, t_2 in be $\geq N$ so that $c \geq N$ and from ③

$$\left|\int_{t_1}^{c} g(x)\,dx\right| < \frac{\epsilon}{2k} \text{ and } \left|\int_{c}^{t_2} f(x)\,dx\right| < \frac{\epsilon}{2k} \rightarrow ④$$

from ① ② ③ & ④, we find a +ve no N such that

$$\left|\int_{t_1}^{t_2} f(x)\,g(x)\,dx\right| \leq |f(t_1)|\left|\int_{t_1}^{c} g(x)\,dx\right| + |f(t_2)|\left|\int_{c}^{t_2} g(x)\,dx\right|$$

$$< k\cdot\frac{\epsilon}{2k} + k\,\frac{\epsilon}{2k} = \epsilon$$

$\forall\, t_1, t_2 \geq N$

Hence by cauchy criterion $\int_a^\infty f(x)\,g(x)\,dx$ is cgt.

Dirichlet, Test

Let
(i) $f : [a, \infty) \rightarrow R$ be bounded, integrable on $[a, t]$ $\forall\, t \in [a\ \infty)$ and $\int_a^t f(x)\,dx$ be bounded in $[a, \infty)$

(ii) $g : [a\ \infty) \rightarrow R$ be monotone bounded on $[a\ \infty)$ such that $\lim_{x \to \infty} g(x) = 0$. Then $\int_a^\infty f(x)\,g(x)\,dx$ is cgt.

OR

Here $\int_a^t f(x)\,dx$ is bounded. It may happen that for different $t \in [a, \infty)$ its values oscillate but remains bounded. So infinite integral $\int_a^\infty f(x)\,dx$ may oscillate. Thus

An infite integral which oscillates finitely becomes cgt after insertion of a bounded monotone factor which tends to zero as $x \to \infty$.

Proof $\because$ g is monotone in $[a\ \infty[$

$\therefore$ It is integrable in $[a\ t]$ $\forall t \geq a$

$\therefore$ It is integrable in $[a\ t]$ $\forall t \geq a$.

Also f is integrable in $[a\ t]$

Therefore by IInd mean value Theorem $\exists$ a no $c \in [t_1\ t_2]$ such that

$$\int_{t_1}^{t_2} f(x)g(x)\,dx = g(t_1)\int_{t_1}^{c} f(x)\,dx + g(t_2)\int_{c}^{t_2} f(x)\,dx \rightarrow (1)$$

$$a \leq t_1 \leq c \leq t_2$$

$\because$ $\int_a^t f(x)\,dx$ is bounded $\forall t \geq a$

$\therefore$ $\exists$ a no K such that

$$\left|\int_a^t f(x)\right| \leq K \qquad \forall t \geq a \rightarrow (2)$$

$$\Rightarrow \left|\int_{t_1}^{c} f(x)\,dx\right| = \left|\int_a^c f\,dx - \int_a^{t_1} f\,dx\right|$$

$$\leq \left|\int_a^c f(x)\,dx\right| + \left|\int_a^{t_1} f(x)\right|$$

$$\leq K + K = 2k \rightarrow (3)$$ $\because a \leq t_1 < c \leq t$ $\because$ By (2)

Similarly

$$\left| \int_c^{t_2} f(x)\,dx \right| \leq 2k \rightarrow (4)$$

Let $\epsilon > 0$ be arbitrary

Since $g(x) \to 0$ as $x \to \infty$, therefor $\exists$ a +ve no N such that

$$|g(x)| < \epsilon/4k \quad \forall x > N$$

In particular for $t_1, t_2 > N$, we have

$$|f(t_1)| < \frac{\epsilon}{4k} \text{ and } f(t_2) < \frac{\epsilon}{4k} \rightarrow (5)$$

using (1), (3), (4) (5), we have

$$\left| \int_{t_1}^{t_2} f(x)\,g(x)\,dx \right| \leq |g(t_1)| \left| \int_{t_1}^{c} f\,dx \right| + |g(t_2)| \left| \int_c^{t_2} f(x) \right|$$

$$< \frac{\epsilon}{4k} \cdot 2k + \frac{\epsilon}{4k} 2k = \epsilon \quad \forall t_1, t_2 > N$$

$\Rightarrow \int_a^\infty f(x)g(x)\,dx$ is Cgt by C-Test

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Example

Prove that $\int_0^\infty \frac{\sin x}{x}\,dx$ Convergent

Solution

$\because \lim\limits_{x\to 0} \frac{\sin x}{x} = 1 \quad \therefore 0$ is not a point of infinite discontinuity

We write

$$\int_0^\infty \frac{\sin x}{x}dx = \int_0^1 \frac{\sin x}{x}dx + \int_1^\infty \frac{\sin x}{x}dx$$

$\int_0^1 \frac{\sin x}{x}dx$ is proper integral

We test $\int_1^\infty \frac{\sin x}{x}dx$

Let $f(x) = \frac{1}{x}$ $\quad g(x) = \sin x$, when $x \geq 1$

$|f(x)| = \frac{1}{x} \leq 1 \quad \forall x \geq 1$

$\Rightarrow$ $f(x)$ is bounded

Now for $x_1 \geq x_2 \geq 1$, we have $\frac{1}{x_1} \leq \frac{1}{x_2}$

i.e $f(x_1) \leq f(x_2)$. $f(x)$ is decreasing

on $[1, \infty)$ and

$$\left|\int_1^t g(x)\,dx\right| = \left|\int_1^t \sin x\,dx\right|$$

$$= |-\cos t + \cos(1)|$$

$$\leq |\cos t| + |\cos(1)| \leq 2$$

$\Rightarrow \int_1^t g(x)\,dx$ is bounded for every $t \geq 1$

Hence by Dirichlet,s Test $\int_1^\infty f(x)g(x)\,dx$

$= \int_1^\infty \frac{\sin x}{x}dx$ is Convergent

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Example

Discuss the convergence (a) $\int_1^{\infty} \sin x^2 \, dx$

(b) $\int_1^{\infty} \cos(x^2)\, dx$

Solutions (a) $\sin(x^2) = \frac{1}{2x} \, 2x \sin x^2$

$$\int_1^{\infty} \sin x^2 \, dx = \int_1^{\infty} \frac{1}{2x} \cdot 2x \sin x^2 \, dx$$

Let $f(x) = 2x \sin x^2 \qquad g(x) = \frac{1}{2x}$

$$\left| \int_1^{t} f(x)\, dx \right| = \left| \int_1^{t} 2x \sin x^2 \, dx \right|$$

$$= \left| -\cos t^2 + \cos(1) \right| \leq 2$$

$\Rightarrow \int_1^{t} f(x)\, dx$ is bounded $\forall\, t \geq 1$

$|g(x)| \leq \frac{1}{2} \quad \forall x \in [1\ \infty)$

$\Rightarrow g$ is bounded and g is decreasing $\forall x \geq 1$

Also $\lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to \infty} \frac{1}{2x} = 0$

Hence by Dirichlet's Test $\int_1^{\infty} f(x)\, g(x)\, dx$ is cgt

(b)

Do yourself.

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Do the best to get to destination.

Example # (Fresnels integrals)

Prove that the integrals (a) $\int_0^\infty \sin(x^2)dx$

(b) $\int_0^\infty \cos(x^2)\,dx$

Converge ~~Conditionally~~ Conditionally

Solutions

We have $\int_0^\infty \sin x^2 = \int_0^1 \sin x^2 dx + \int_1^\infty \sin x^2 dx$

1st integral $\int_0^1 \sin x^2 dx$ is proper

We now test $\int_1^\infty \sin x^2 dx$ for Convergence at ∞

$$\int_1^\infty \sin x^2 dx = \int_1^\infty (2x \sin x^2)\cdot\frac{1}{2x}\, dx$$

Let $f(x) = 2x \sin x^2 \qquad g(x) = \frac{1}{2x}$

$g(x)$ is monotone and $\to 0$ as $x \to \infty$

$$\left|\int_1^t f(x)dx\right| = \left|\int_1^t \sin x\, dx\right| = |\cos 1 - \cos t|$$
$$\leq |\cos 1| + |\cos t| \leq 2$$

$\Rightarrow \int_1^t f(x)dx$ is bounded $\forall\, t \geq 1$. Hence by Dirichlet test $\int_1^\infty f(x)g(x)dx = \int_1^\infty \frac{\sin x}{x} dx$ is cgt. Thus $\int_0^\infty \sin x^2 dx$ is cgt.

Now we check absolute Convergence

let $t_1, t_2 > 0$

$\int_{t_1}^{t_2} \sin x^2 dx$

let $x^2 = u \Rightarrow x = \sqrt{u}$

$2x\,dx = du$

$dx = \frac{du}{2x} = \frac{du}{2\sqrt{u}}$

$$\int_0^\infty |\sin x^2|\,dx = \frac{1}{2}\int_0^\infty \left|\frac{\sin u}{\sqrt{u}}\right| du$$

$$= \frac{1}{2}\int_0^1 \frac{|\sin u|}{\sqrt{u}}\,du + \int_1^\infty \frac{|\sin u|}{\sqrt{u}}\,du$$

$$= \text{finite} + \infty \text{ (proved already)}$$

$\Rightarrow \int_0^\infty |\sin x^2|\,dx$ diverges

Thus $\int_0^\infty \sin x^2\,dx$ Converges conditionally

(b)

$$\int_0^\infty \cos(x^2)\,dx = \int_0^1 \cos x^2\,dx + \int_1^\infty \cos x^2\,dx$$

1st integral is proper

$$\int_1^\infty \cos x^2\,dx = \int_1^\infty 2x \cos x^2 \cdot \frac{1}{2x}\,dx$$

Let $f(x) = 2x\cos x^2 \quad g(x) = \frac{1}{2x}$

$g(x)$ is $\downarrow$ and $g \to 0$ as $x \to \infty$

$\left|\int_1^t f(x)\,dx\right| \leq 2 \quad t \geq 1$

$\Rightarrow \int_1^t f(x)\,dx$ is bounded for all $t > 1$

$\Rightarrow \int_1^\infty f(x)\,g(x)\,dx$ is cgt. Check absolute convergence yourself

Example

Show that $\int_0^\infty e^{-ax} \frac{\sin x}{x} dx \quad a \geq 0$

is convergent

Solution

Let $f(x) = \left(\frac{\sin x}{x}\right)$ and $g(x) = e^{-ax} = \frac{1}{e^{ax}}$

As proved above $\int_0^\infty f(x) dx = \int_0^\infty \frac{\sin x}{x} dx$

is cgt. Also $g(x)$ is bounded and monotonically decreasing function $\forall x > 0$

$\Rightarrow$ By Abel's Test $\int_0^\infty f(x) g(x) dx = \int_0^\infty e^{-ax} \frac{\sin x}{x} dx$

is cgt

Example

If $a \neq 0$, then prove that $\int_0^\infty e^{-a^2x^2} \sin bx\, dx$ is absolutely cgt

Solution

We have

$$\int_0^\infty \left| e^{-a^2x^2} \sin bx \right| dx = \int_0^1 \left| e^{-a^2x^2} \sin bx \right| dx + \int_1^\infty \left| e^{-a^2x^2} \sin bx \right| dx$$

1st integral is proper and hence is absolutely cgt since if f is R-integrable, then $|f|$ is also R-integrable

Also $\int_1^t \left| e^{-a^2x^2} \sin bx \right| dx \leq \int_1^t e^{-a^2x^2} dx \quad \forall t > 0$

We check convergence of $\int_1^{\infty} e^{-a^2x^2}\,dx$

let $f(x) = e^{-a^2x^2}$ $g(x) = \frac{1}{x^2}$

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{e^{-a^2x^2}}{1/x^2} = \lim_{x\to\infty}\frac{x^2}{e^{a^2x^2}} \quad \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty}\frac{2x}{2a^2x\,e^{a^2x^2}} = \lim_{x\to\infty}\frac{1}{a^2x\,e^{a^2x^2} + 2a^2x\,e^{a^2x^2}}$$

$= 0$

$\because \int_1^{\infty}\frac{1}{x^2}\,dx$ is cgt $\therefore \int_1^{\infty} e^{-a^2x^2}\,dx$ is cgt

Hence by comparison test $\int_1^{\infty}|e^{-a^2x^2}\sin bx|\,dx$ is cgt. Thus $\int_0^{\infty} e^{-a^2x^2}\sin bx\,dx$ is absolutely cgt

---

Example

Show that $\int_0^{\infty} e^{-x}\cos x\,dx$ is absolutely convergent

Solution

$\because |e^{-x}\cos x| \leq e^{-x} \quad \forall x\in[0,\infty)$

and $\int_0^{\infty} e^{-x}\,dx$ is cgt

Therefore by Comparison test $\int_0^{\infty} e^{-x}\cos x\,dx$ is absolutely convergent

Example

Examine the Convergence of $\int_0^\infty \frac{x}{1+x^2}\sin x\,dx$

Solutions

Let $f(x) = \sin x \qquad g(x) = \frac{x}{1+x}$

Then $\left|\int_1^t f(x)\right| \le 2$

$\Rightarrow \int_1^t f(x)$ is bounded for $t \geq 1$

Also $\lim_{x\to\infty} \frac{x}{1+x^2} = 0$

$g(x)$ is bounded and monotonically decreasing in $[1, \infty)$

$\therefore$ By Dirichlet test $\int_1^\infty f(x)g(x)dx = \int_1^\infty \frac{x}{1+x^2}\sin x\,dx$ Converges

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Example

Examine the Convergence $\int_1^\infty \frac{\cos x\,dx}{\sqrt{x+x^2}}$

Solution

Let $f(x) = \cos x$ & $g(x) = \frac{1}{\sqrt{x+x^2}}$

Then $\left|\int_1^t \cos x\,dx\right| \le 2 \Rightarrow \int_1^t \cos x\,dx$ is bounded for $t \geq 1$. Also $g(x)$ is bounded and $\downarrow$

$\lim_{x \to \infty} g(x) = 0$

$\therefore$ By Dirichlet's Test $\int_1^\infty f(x) g(x) dx = \int_1^\infty \frac{\cos x}{\sqrt{x+x^2}} dx$

is cgt

Example

Examine the convergence of $\int_a^\infty \frac{\cos \alpha x - \cos \beta x}{x} dx$, $a > 0$

Solution

$$\int_a^\infty \frac{\cos \alpha x - \cos \beta x}{x} dx = \int_a^\infty \frac{\cos \alpha x}{x} dx - \int_a^\infty \frac{\cos \beta x}{x} dx$$

$$= I_1 - I_2 \text{ (say)}$$

Consider

$$I_1 = \int_a^\infty \frac{\cos \alpha x}{x} dx$$

Let $f(x) = \cos \alpha x$ $\qquad g(x) = \frac{1}{x}$

$$\left|\int_a^t f(x) dx\right| = \left|\int_a^t \cos \alpha x \, dx\right| = \left|\frac{\sin \alpha t - \sin \alpha a}{\alpha}\right|$$

$$\leq \frac{1}{|\alpha|} + \frac{1}{|\alpha|} = \frac{2}{|\alpha|}$$

$\therefore \int_a^t f(x) dx$ is bounded $\forall t \geq a$

$\therefore$ $g(x)$ is bounded and monotonically decreasing tends to 0 as $x \to \infty$

$\therefore$ By Dirichlet's Test $\int_a^{\infty} f(x)\, g(x)\, dx$

$$I_1 = \int_a^{\infty} \frac{\cos \alpha x}{x}\, dx \text{ is cgt}$$

Similarly $I_2 = \int_a^{\infty} \frac{\cos \beta x}{x}\, dx$ is cgt

Hence the given integral converges

Example

Test the convergence of $\int_a^{\infty} \frac{\sin x \log x}{x}\, dx \quad a>0$

Solution

Let $f(x) = \sin x \quad g(x) = \frac{\log x}{x}$

$x \geq a > 0$

Then $\left|\int_a^t f(x)\, dx\right| = \left|\int_a^t \sin x\, dx\right| = |\cos a - \cos t|$

$\leq 1+1 = 2 \quad \forall t \geq a$

$\Rightarrow \int_a^t f(x)\, dx$ is bounded $\forall t \geq a$

Also $g'(x) = \frac{1-\log x}{x^2} < 0 \quad \forall x > e$

$\Rightarrow g(x)$ is decreasing on $]e \;\; \infty[$

$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{\log x}{x} = \lim_{x\to\infty} \frac{1}{x} = 0$

$\therefore$ $g(x)$ is bounded [infact $0 \leq \frac{\log x}{x} \leq \frac{1}{e}$] and monotonically tending to zero as $x \to \infty$

$$\Rightarrow \int_e^{\infty} f(x)g(x)\,dx = \int_e^{\infty} \frac{\sin x \log x}{x}\,dx \text{ is cgt}$$

Now $\int_a^{\infty} f(x)\,dx = \int_a^{e} f(x)\,dx + \int_e^{\infty} f(x)\,dx$

Bound because $\int_a^t f(x)\,dx$ is bounded for all $t \geq a$

$$\Rightarrow \int_a^{\infty} f(x)\,dx \text{ is cgt}$$

<u>Example</u> #

Show that (i) $\int_0^{\infty} e^{-px} \cos qx \, dx \quad p>0$

(ii) $\int_0^{\infty} \frac{\sin px}{(1+e^{x})(1+e^{-x})}\,dx \quad (p\neq 0)$

are absolutely cgt

(i) Let $f(x) = e^{-px} \quad (p>0) \quad \forall x>0$

$g(x) = \cos qx$

For all $t>0$, f is bounded and integrable $[0,t]$

As $e^{-px} > 0 \quad \forall x>0$

$$\lim_{t\to\infty} \int_0^t e^{-px}\,dx = \lim_{t\to\infty} \frac{1}{p}\left(1-e^{-pt}\right)$$

$$= \frac{1}{p} \quad (\because p>0), \text{ a finite no}$$

So $\int_0^{\infty} e^{-px}\,dx$ is cgt

Since $e^{-px} > 0 \quad \forall x \geq 0$

So $\int_0^\infty f(x)dx$ converges absolutely

$\because \quad -1 \leq \cos qx \leq 1 \quad \forall x \geq 0$

$\therefore \quad g(x)$ is bounded for all $x \geq 0$

$\int_0^t g(x)dx = \frac{\sin qt}{t}$ i.e g is integrable on $[0, t] \ \forall t \geq 0$

Thus by theorem of absolute convergence of the integral of product of functions

$\int_0^\infty f(x)g(x)dx$ is absolutely cgt

(ii)

Let $f(x) = \frac{1}{(1+e^x)(1+e^{-x})} \quad x \geq 0$

$g(x) = \sin px \quad (p \neq 0)$

Then $f(x) > 0 \quad \forall x \geq 0$

$\forall t \geq 0$

$$\int_0^t f(x)dx = \int_0^t \frac{e^x}{(e^x+1)^2}dx$$

$$= -\left[\frac{1}{1+e^x}\right]_0^t = \frac{1}{2} - \frac{1}{e^t+1}$$

$\lim_{t \to \infty} \int_0^t f(x)dx = 1/2$. Hence $\int_0^\infty f(x)$ is absolutely cgt

Now $|g(x)| \leq 1 \quad \forall x \geq 0$

and $\forall t > 0$

$$\left| \int_0^t g(x)\,dx \right| = \frac{1}{|p|} \left| 1 - \cos pt \right| \leq \frac{2}{|p|}$$

, a finite number.

$\Rightarrow \int_0^t g(x)\,dx$ is bounded $\forall t > 0$.

Therefore by theorem of absolute convergen of the integral of the product of functions

$\int_0^\infty f(x) g(x)\,dx$ is absolutely cgt

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Example

Show that the integral $\int_1^\infty \frac{\log x}{x^p}\,dx$ is cgt for $p > 1$ and is dgt for $p \leq 1$

Solution

$$f(x) = \frac{\log x}{x^p} \geq 0 \qquad \forall x \geq 1$$

for $\lambda > 0$, consider

$$x^{p-\lambda} f(x) = \frac{\log x}{x^\lambda}$$

$$\lim_{x \to \infty} x^{p-\lambda} f(x) = \lim_{x \to \infty} \frac{\log x}{x^\lambda} \quad \left(\frac{\infty}{\infty}\right)$$

$$= 0 \qquad \text{Lo-Hospital Rule}$$

So if $P-\lambda > 1$, then by μ- test

$$\int_1^\infty \frac{\log x}{x^P} \text{ is cgt}$$

i.e $\int_1^\infty \frac{\log x}{x^P}dx$ is cgt if $P > 1+\lambda$ $\lambda>0$

i.e $P>1$ $\because \lambda>0$

Again

Since $\lim_{x\to\infty} x^P f(x) = \lim_{x\to\infty} \log x = +\infty$

So by μ test $\int_1^\infty \frac{\log x}{x^P}$ is dgt if $P\leq 1$

<u>OR</u>

Let $f(x) = \frac{\log x}{x^P}$, $g(x) = \frac{1}{x^P}$

$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \log x = \infty$

and $\int_1^\infty f(x)$ is dgt if $\int_1 g(x)dx$ is dgt

Now $\int_1^\infty \frac{1}{x^P}dx$ is dgt for $P\leq 1$

Therefore $\int_1^\infty \frac{\log x}{x^P}$ is dgt for $P\leq 1$

<u>Example</u>

Show that $\int_0^\infty \frac{\sin mx}{x}dx$ is cgt for $m>0$

<u>Solution</u>

$\because \lim_{x\to 0} \frac{\sin mx}{x} = \lim_{x\to 0} \frac{\sin mx}{mx}\cdot m = m$, finite

So $\int_0^1 \frac{\sin mx}{x}\,dx$ is proper and hence

Cgt. Thus

$$\int_0^\infty \frac{\sin mx}{x}\,dx = \int_0^1 \frac{\sin mx}{x}\,dx + \int_1^\infty \frac{\sin mx}{x}\,dx$$

We check the Convergence of $\int_1^\infty \frac{\sin mx}{x}\,dx$

Let $f(x) = \sin mx \qquad g(x) = \frac{1}{x}$

g is monotone decreasing and bounded.

in $[1\ \infty[$ and $\lim_{x\to\infty} g(x) = 0$

For all $t \geq 1$

$$\left|\int_1^t f(x)\,dx\right| = \frac{1}{m}\left|\cos m - \cos mt\right|$$

$$\leq \frac{2}{m} \qquad \because |\cos x| \leq 1$$

$\Rightarrow \int_1^t f(x)\,dx$ is bounded for all $t \geq 1$

Therefore by Dirichlet test $\int_1^\infty f(x)g(x)\,dx$

$= \int_1^\infty \frac{\sin mx}{x}\,dx$ is cgt

Hence $\int_0^\infty \frac{\sin mx}{x}\,dx$ is cgt.

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Example

Show that $\int_1^\infty x^k \left(\frac{x + \sin x}{x - \sin x} \right) dx$ is cg only when $k < -1$

Solution

Let $f(x) = x^k \left(\frac{x + \sin x}{x - \sin x} \right)$

and $g(x) = x^k = \frac{1}{x^{-k}}$

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{x + \sin x}{x - \sin x}$$

$$= \lim_{x \to \infty} \frac{1 + \frac{\sin x}{x}}{1 - \frac{\sin x}{x}} = \frac{1 + \lim_{x \to \infty} \frac{\sin x}{x}}{1 - \lim_{x \to \infty} \frac{\sin x}{x}}$$

$$= \frac{1+0}{1-0} = 1$$

$\because$ $\sin x$ is bounded and $\frac{1}{x} \to 0$ as $x \to \infty$

$\therefore$ $\lim_{x \to \infty} \frac{\sin x}{x} = 0$

By limit Comparison test both integrals $\int_1^\infty f(x)\,dx$ and $\int_1^\infty g(x)\,dx$ behave alike.

Now $\int_1^\infty g(x) = \int_1^\infty \frac{1}{x^{-k}}$ is cgt only if $-k > 1$

i.e if $k < -1$

Therefore $\int_1^\infty f(x)\,dx = \int_1^\infty x^k \left(\frac{x + \sin x}{x - \sin x} \right) dx$

is Convergent only if $k < -1$. Proved

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Example

Show that $\int_2^\infty \frac{1}{x^k \log x}$ converges for $k>1$ and diverges for $k \le 1$

$$g(x) = \frac{1}{x^k}$$

$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1}{\log x} = 0$, a finite no

Also $\int_2^\infty g(x)\,dx$ is cgt if $k>1$

Hence if $\int_2^\infty g(x)\,dx$ is cgt, then $\int_2^\infty \frac{1}{x^k \log x}\,dx$ is cgt

Note that for $k \le 1$, we can not take result because $\lim_{x\to\infty} \frac{f}{g} = 0$

For $k=1$

Integral $= \int_2^\infty \frac{1}{x\log x}\,dx$, which is dgt

as $\int_2^t \frac{1}{x\log x}\,dx = \log(\log x)\Big]_2^t$

$= \log(\log t) - \log(\log 2)$

$\lim_{t\to\infty} \int_2^t \frac{1}{x\log x}\,dx = \log(\infty) = \infty$

$\Rightarrow$ Integral diverges

P.T.O

For $k<1$, $x^k < x \quad \forall x \geq 2$

Hence $f(x) = \dfrac{1}{x^k \log x} > \dfrac{1}{x \log x} \quad \forall x \geq 2$, $k<1$

Since $\displaystyle\int_2^\infty \frac{1}{x \log x}$ is dgt, therefor by Comparison test $\displaystyle\int_2^\infty \frac{1}{x^k \log x}$ is dgt

Example

Test for Convergence $\displaystyle\int_1^\infty \sin x^p dx$

Solution

$$\int_1^\infty \sin x^p dx = \int_1^\infty \frac{1}{px^{p-1}} px^{p-1} \sin x^p dx$$

Let $f(x) = px^{p-1} \sin x^p \quad g(x) = \dfrac{1}{px^{p-1}}$

For $p>1$ $g(x)$ is monotone decreasing for $x \geq 1$ and $\to 0$ as $x \to \infty$

Also $\left| \displaystyle\int_1^t px^{p-1} \sin x^p dx \right| = \left| -\cos t^p + \cos 1 \right| \leq 2 \quad \forall t \geq 1$

$\Rightarrow \displaystyle\int_1^t px^{p-1} \sin x^p dx$ is bounded for all $t \geq 1$

Hence by Dirichlet's test $\displaystyle\int_1^\infty \frac{1}{px^{p-1}} px^{p-1} \sin x^p dx$

$= \displaystyle\int_1^\infty \sin x^p dx$ is cgt for $p>1$

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